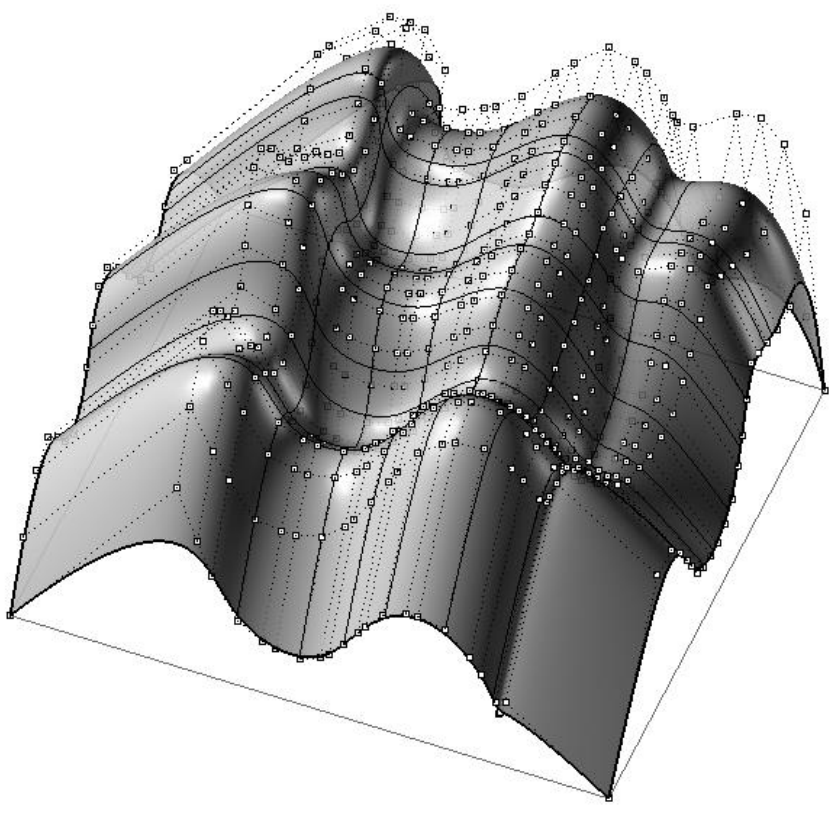
TEXTBOOK

- Lines and surfaces -

Text supporting the course Geometrical and Generative Modelling



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1. GENERAL NOTES ON LINES AND SURFACES

The POINT is a geometrical entity without dimension.

The LINE is a one-dimensional entity generated by the continuous movement of the point.

Lines can be CURVED or non-curved; non-curved lines are called STRAIGHT LINES.

Each straight line has a DIRECTION; Direction is the common property of a family of parallel lines.

Each straight line contains an IMPROPER POINT, that is, a point at infinity. To each direction of lines corresponds only one improper point, that is, all parallel lines to each other have the same point of infinity, hence it is said that parallel lines are lines that intersect each other at infinity.

SURFACE is a two-dimensional entity generated by the continuous movement of the line.

The GENERATRIX is the line, deformable or undeformable, that moves in space to generate the surface.

The DIRECTRIX is the line or surface on which the generatrix rests in its movement.

If the directrix is a surface, then the generated surface is said to have a CORE.

When a straight generatrix moves continuously in space, retaining the direction, supported by a straight directrix with a direction different from its own, the PLAN is generated.

Each plan has an ORIENTATION; orientation is the common property of a family of parallel planes.

Each plane contains an IMPROPER LINE, that is, a line at infinity.

To each orientation of planes corresponds only one improper line, that is, all planes parallel to each other have the same line of infinity, hence it is said that parallel planes intersect at infinity.

An orientation contains a multitude of directions.

The *locus* of all improper points and all improper lines is the IMPROPER PLAN, that is, the plane of infinity.

When a surface can be generated by moving a straight line it is said to be RULED.

When a surface cannot be generated by moving a straight line, it is said to be CURVED.

The ORDER[1] of an algebraic surface, that is, the order or degree of the polynomial that defines it, can be interpreted as the maximum number of points at which a line can intersect it. For example, the plane is a 1st order surface and a spherical surface is a 2nd order surface.

When a ruled surface can be "unrolled" to a plane without causing "folds" or "tears" the surface is said to be DEVELOPABLE; only ruled surfaces can be developable, although not all ruled surfaces are.

2. PLANAR LINES

There is no single criterion for classifying lines as flat or spatial. However, several criteria can be used to group them in families. Different criteria can lead to identical lines in different families. Or a one may lead to the exclusion of specific lines.

2.1. Conic lines

A CONICAL line results from the intersection produced by a plane on a conical surface (Figure 1) and can be represented by a second degree polynomial[2] as follows:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad (1)$$

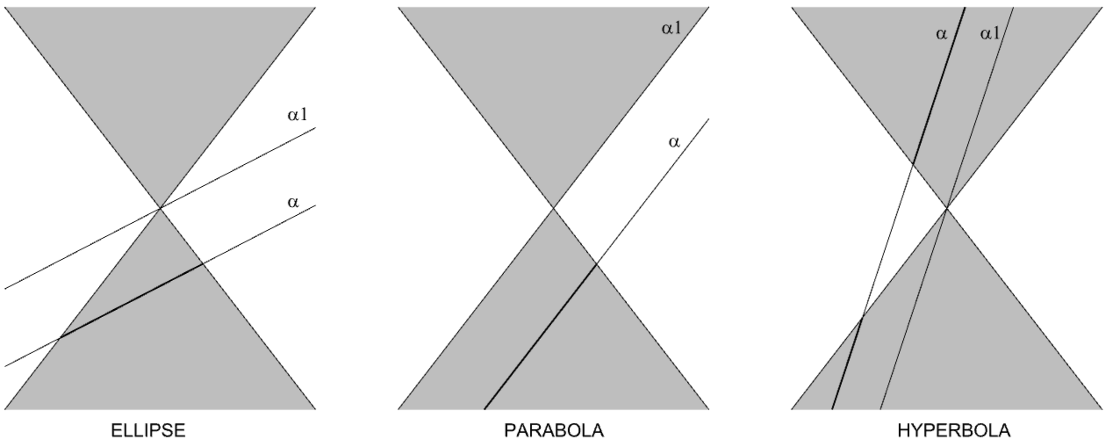

figure 1: Conic line according to the relative position of the sectioning plane and the conical surface.

The conic lines are of three types: ellipse, parabola and hyperbola.

The ellipse is produced when the plane intersects all the generatrices of the conical surface.

The parabola is produced when the plane is parallel to one generatrix of the conical surface.

[1] Note that this definition of order is not the same as will be discussed later when it comes to Splines

[2] This expression can be generalized to higher degrees giving rise to other types of lines (cubic, quartic,…).

The hyperbola is produced when the plane is parallel to two generatrices of the conical surface.

2.1.1. Ellipse

The ellipse (of which the circumference is a particular case) is defined as the flat curve in which all points meet the condition that their summed distances to two fixed points called foci is constant.

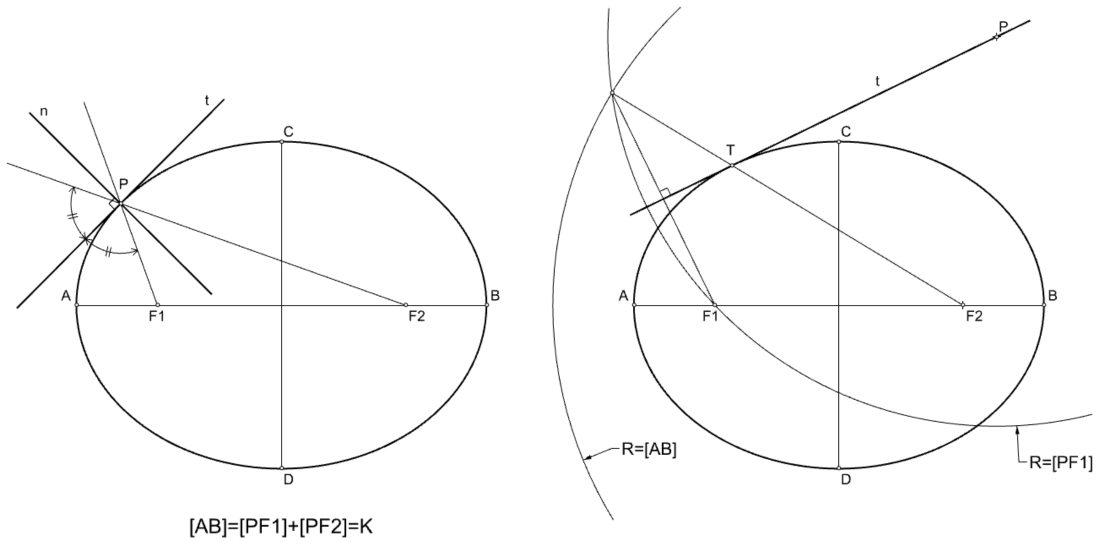

figure 2: Definition, tangent and normal (left). Tangent line from an exterior point (right).

The ellipse can be obtained by an affine transformation of a circumference (figure 4), and is defined by two conjugated diameters (corresponding to a circumscribed parallelogram tangent to the ellipse at the midpoints of the sides). When two diameters are conjugated, one bisects all chords parallel to the other and vice versa.

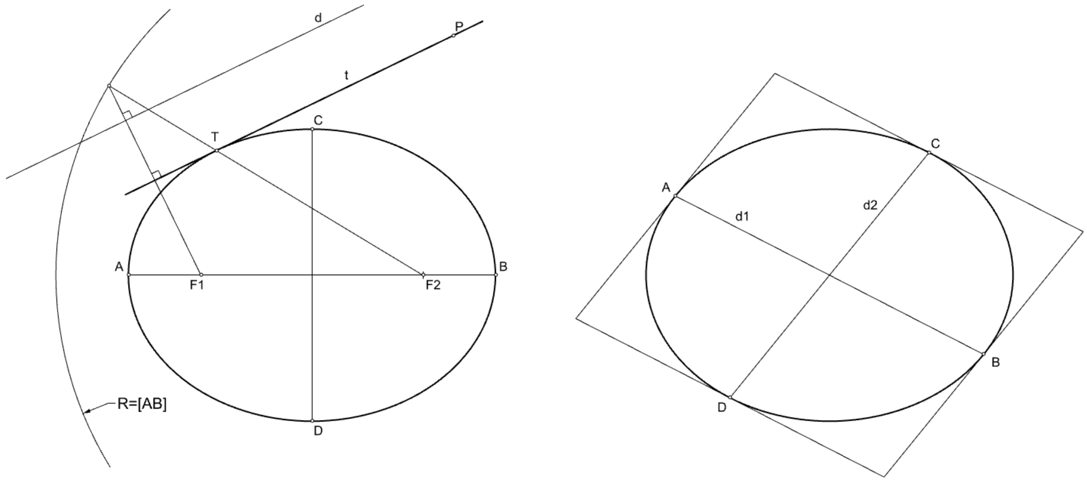

figure 3: Tangent with a given direction (left). Ellipse defined by two conjugated diameters d1 and d2 (right).

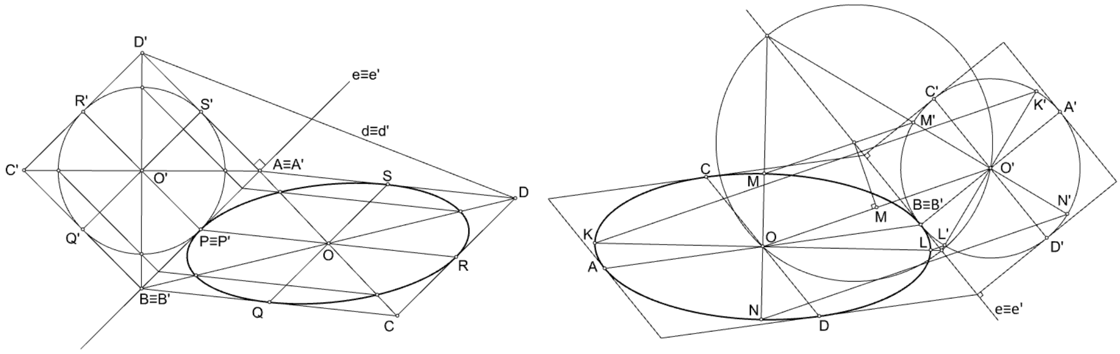

figure 4: Affine transformation between ellipse and circumference with the determination of the principal axis.

2.1.2. Parabola

A parabola is a flat curve that is defined by the condition that for any point on the curve, the distance to a fixed point, called focus, and the distance to a fixed line, called directrix, is equal.

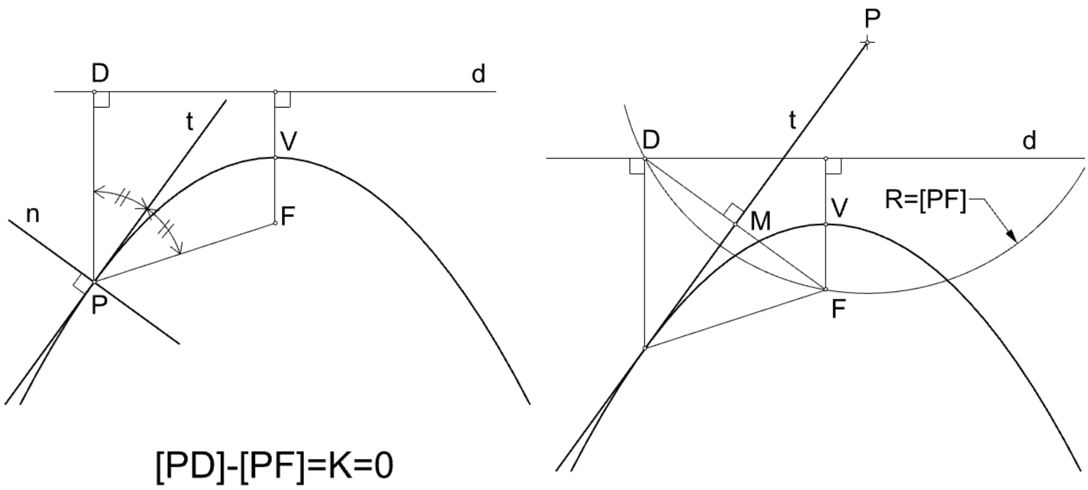

figure 5: Definiton, tangent and normal (left). Tangent from a given point.

The parabola can be defined by two tangents and the respective points of tangency as it can be seen in the next figure. In that case the points of the parabola can be defined recursively.

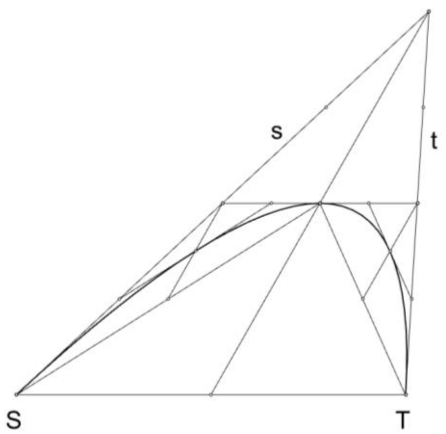

figure 6: Parabola defined by two tangent lines and the corresponding tangent points.

2.1.3. Hyperbola

The hyperbola is a flat curve that is defined similarly to the ellipse, but with the distinction that the difference of the distances from a point on the curve to the foci is constant.

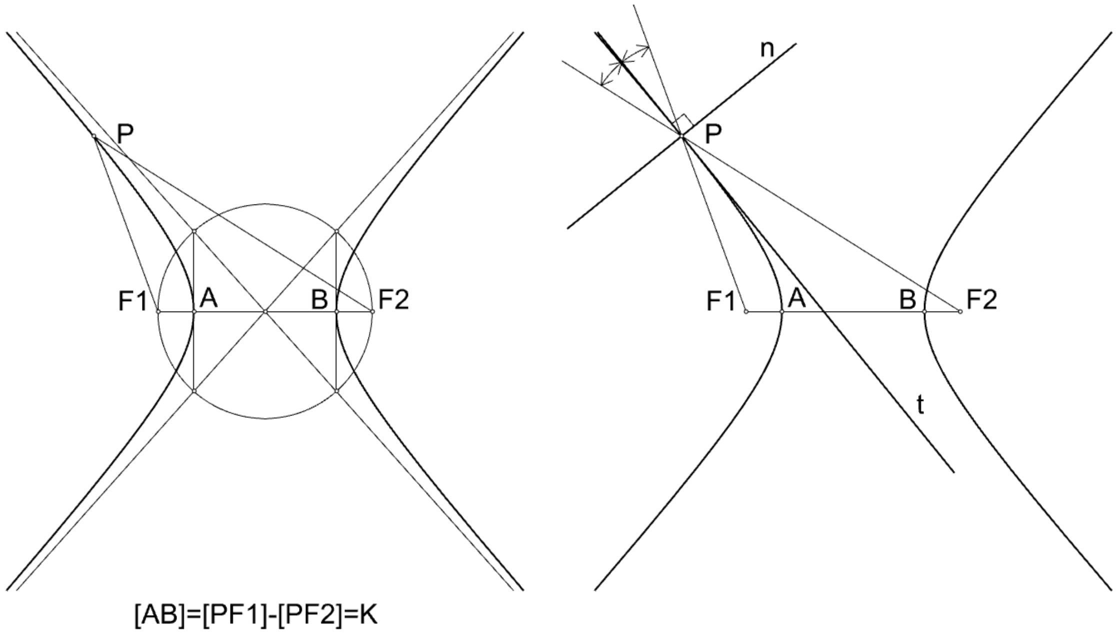

figure 7: Definition (left). Tangent from a point on the curve, tangent and normal (right).

2.1.4. Free curves

From a practical point of view, it is important to have a type of line that can represent those lines that can be called free lines. Free lines are those that result from a spontaneous drawing gesture. These lines do not have a geometric definition a *priori*. However, when representing them on a computer, it is necessary to rationalize them in some way. The simplest way to rationalize them, although usually resulting in unattractive practical result, is to represent them through a sequence of ARCS OF CIRCUMFERENCE, as shown in figure 8. This way of rationalizing a free line originates from the analogical drawing as it can be understood by the implicit economy of means.

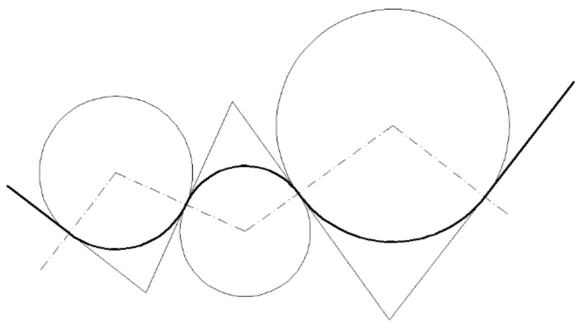

figure 8: Rationalizing a line trough a sequence of arcs of circumference.

2.2. Splines

One way to designate freeform lines is by using the term SPLINE. This term goes back to the use of wooden strips that were bent and curved to generate molds for aeronautical construction[3]. In geometric terms, a spline is a smooth curve based on the joining of one or more curves defined by polynomial functions that are adjacent at points called INTERNAL KNOTS (K)[4]. The endpoints of a spline are also called KNOTS..

2.2.1. Curva de Bézier

The simplest spline is the BÉZIER CURVE[5]. A Bezier curve is a B-spline (see next section) without Internal Knots (K). A Bézier curve is defined by CONTROL POINTS (C) which are in precise relationship with its DEGREE (D):

$$C = D + 1 \qquad (2)$$

In figure 9 we have three Bézier curves, with degree 2, 3 and 4, from left to right, respectively. Note that a degree 1 Bezier curve is no more than a STRAIGHT LINE SEGMENT.

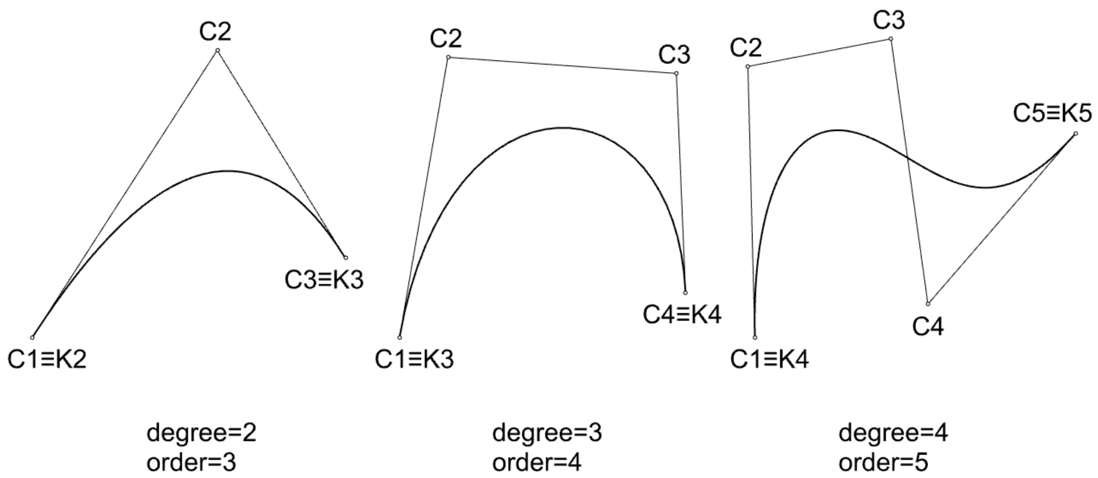

figure 9: Bézier curves of degree 2, 3 and 4.

In the curves shown in figure 9, the knot vectors are, respectively, K={0.0,0.0,1.0,1.0}, K={0.0,0.0,0.0,1.0,1.0,1.0} and K={0.0,0.0, 0.0.0.01.0,1.0,1.0,1.0}. Note that the number of

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[3] See Wikipedia article (http://en.wikipedia.org/wiki/Spline_%28mathematics%29).

[4] The internal knots correspond to the points at which the various polynomial curves are adjacent in geometric space. Spline knots correspond to a knot vector in parametric space. When speaking of uniform node spacing, in the designation of uniform B-Splines, reference is made to spacing in parametric space rather than geometric space. In this regard see the article on B-Splines at Wolfram MathWorld (http://mathworld.wolfram.com/B-Spline.html).

[5] See Wikipedia article (http://en.wikipedia.org/wiki/B%C3%A9zier_curve).

knots is twice the degree of the curve. Another important rule is that the first geometric point of the curve corresponds to the node whose index equals the degree. For example, on grade two curve the second node is 0.0 and this node corresponds to the first endpoint of the curve. Similarly, if we follow the knot vector in reverse order, we identify the knot corresponding to the second endpoint of the curve. Still as an example, in a Bézier of degree 2, the geometric points of the curve correspond, in the parametric space, to the values between parameter K2 and K3, that is, between 0.0 and 1.0. It is further important to note that equal spacing in parametric space does not correspond to equal spacing in geometric space.

Another important concept is the ORDER (O) of the curve. By definition:

$$O = D + 1 \qquad (3)$$

De Casteljau's algorithm allows us to understand the graphical construction of a Bézier curve whatever its degree. In figure 9 one can see the logic of the graphical construction of a Bezier curve of degree 3. It is important to realize that this logic can be generalized to any degree.

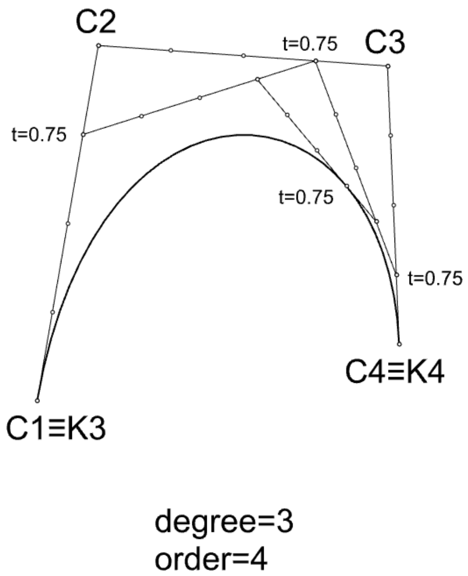


figure 10: De Casteljau algorithm for the construction of a Bézier curve of degree 3.

The figure illustrates the construction of a point given by the parameter t = 0.75. The parameter range for the construction of the Bézier curve is in this case given by the interval [0.0,1.0] since this is the interval corresponding to nodes K3 and K4. Parameter point 0.0 is point K3 and parameter point 1.0 is point K4, which coincides with the first point and last control point, respectively. Since the degree is 3, the algorithm implies 3 subdivision levels for constructing curve points. The number of subdivision levels is equal to the degree of the curve.

An interesting point to notice is that a Bézier curve of degree 2 is a parabola, ie, it is a particular case of a conic line. Another interesting aspect to notice about the degree of the Bezier curve is that it is also related to the number of possible inflections in it. Thus, the number of possible inflections (I) is equal to:

$$I = D - 2 \qquad (4)$$

It is important to note that a Bézier curve cannot be used for the representation of conics in general, only parabolas.

2.2.2. B-Splines

One way to expand the use of Bézier curves is by using B-SPLINES (Basis Spline). In practice, a B-Spline is a smooth arrangement of Bézier curves. In other words, at the transition points (corresponding to the internal knots), the Bezier curves share the same tangent lines. A B-Spline can be defined in two ways: through control points or through point interpolation. When a B-Spline is defined using control points, the curve does not pass through those points (although the knot vector can be defined so that the curve passes through the endpoints). When it is defined by point interpolation the curve passes through those points. Note that the interpolated points are not the internal nodes. In practice, B-Splines give us a way to construct Bézier curves that smoothly articulate with each other. B-Splines contain internal knots, that is, points at which two consecutive Bezier curves contact each other. If a B-Spline contains an internal knot, it means that it is composed of two Bezier curves, and so on.

Let us return to the question of the knot vector, and its relation to the curve configuration, through examples.

In figure 11 we have a B-Spline of degree 3 whose knot vector is K={0.0,1.0,2.0,3.0,4.0,5.0,6.0,7.0,8.0}. Since the degree is three, the first endpoint corresponds to knot K3=2.0 and the last endpoint corresponds to node K7=6.0. This knot vector has all elements equally spaced and so is called UNIFORM. If the elements of the vector are not equally spaced, it is said NOT UNIFORM. However, it turns out that the endpoints of the curve do not coincide with the first and last control points. Analysing the internal knots, we find that this curve is, in practice, an arrangement of 4 Bezier curves "glued" at the points corresponding to K4, K5 and K6.

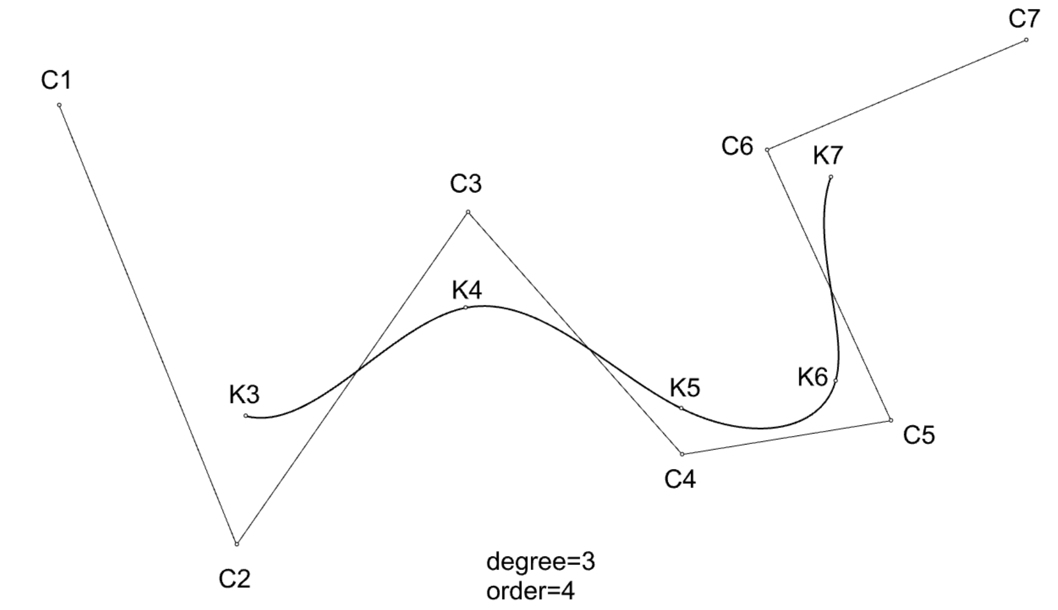


figure 11: B-Spline of degree 3 whose knot vector is K={0.0,1.0,2.0,3.0,4.0,5.0,6.0,7.0,8.0}.

In any case, the number of elements (N) of the knot vector is given by the expression:

$$N = C + O - 2 \qquad (5)$$

In the example of figure 12, we have B-Spline of degree 3 whose knot vector is K={0.0,0.0,0.0,1.0,2.0,3.0,4.0,5.0,6.0}. In this case, it is found that the first endpoint of the curve coincides with the first control point. Also this curve is an arrangement of 4 Bézier curves.

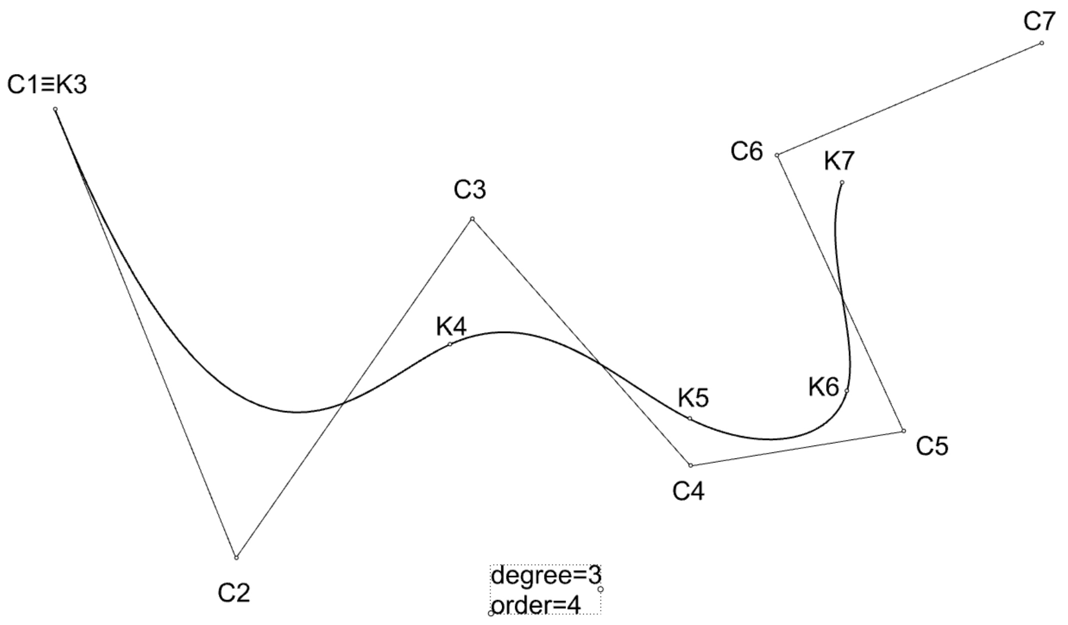


figure 12: B-Spline of degree 3 whose knot vector is K={0.0,0.0,0.0,1.0,2.0,3.0,4.0,5.0,6.0}.

In Figure 13 we have a B-Spline of degree 3 whose node vector is K={0.0,0.0,0.0,1.0,2.0,3.0,4.0,4.0,4.0}. In this case it turns out that the endpoints of the curve coincide with the first point and last control point. This corresponds to the most commonly B-Splines used.

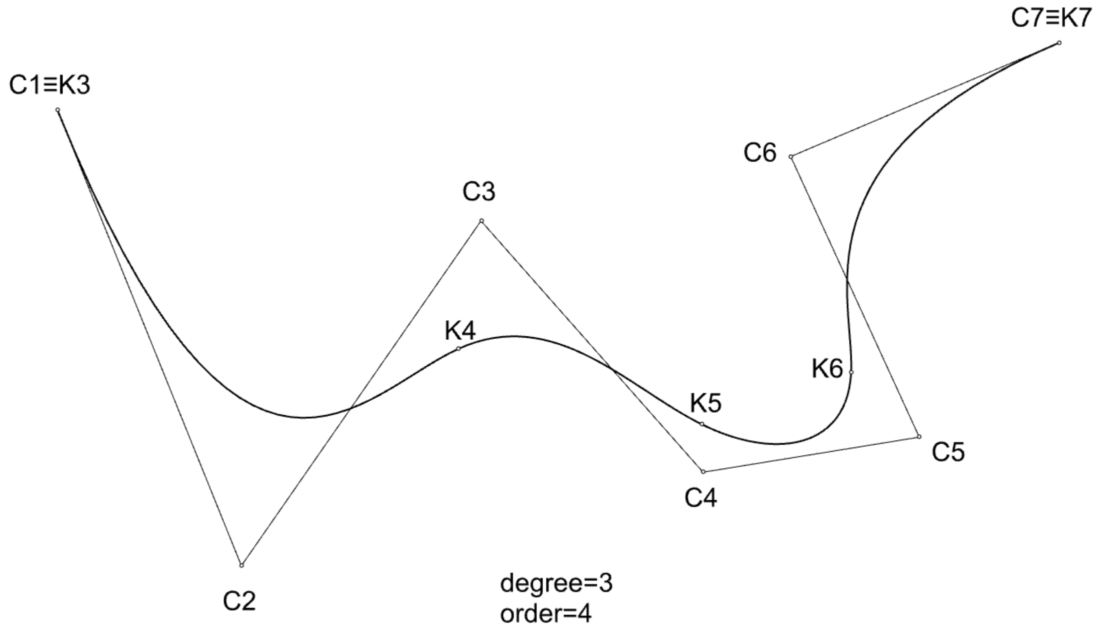


figure 13: B-Spline of degree 3 whose node vector is K={0.0,0.0,0.0,1.0,2.0,3.0,4.0,4.0,4.0}.

Both Bézier curves and their generalization given by B-Splines are INVARIANT UNDER AFFINE TRANSFORMATION, that is, if we subject the control points of a B-Spline to an affine transformation, the transformed points are control the points of a B -Spline affine to the first one as illustrated in figure 14.

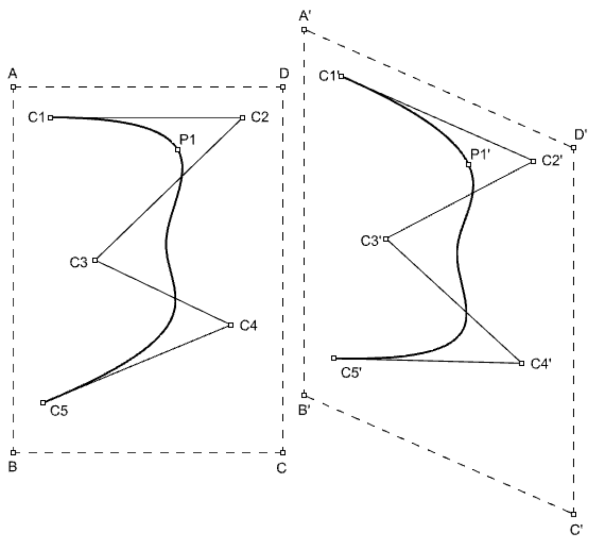


figure 14: Two affine B-SPLINES.

It should be recalled that an affine transformation transforms a parallelogram [ABCD] into another parallelogram [A'B'C'D’]. Note that the point P1' is related to the point P1, and it lies in the B-Spline affine to the first one.

2.2.3. NURBS Lines

However, B-Splines are not invariant under a PROJECTIVE TRANSFORMATION. That is, subjecting the control points of a B-Spline to a projective transformation, the homologous points of the control points are control points of a B-Spline which is not affine with the first B-Spline, as shown in figure 15.

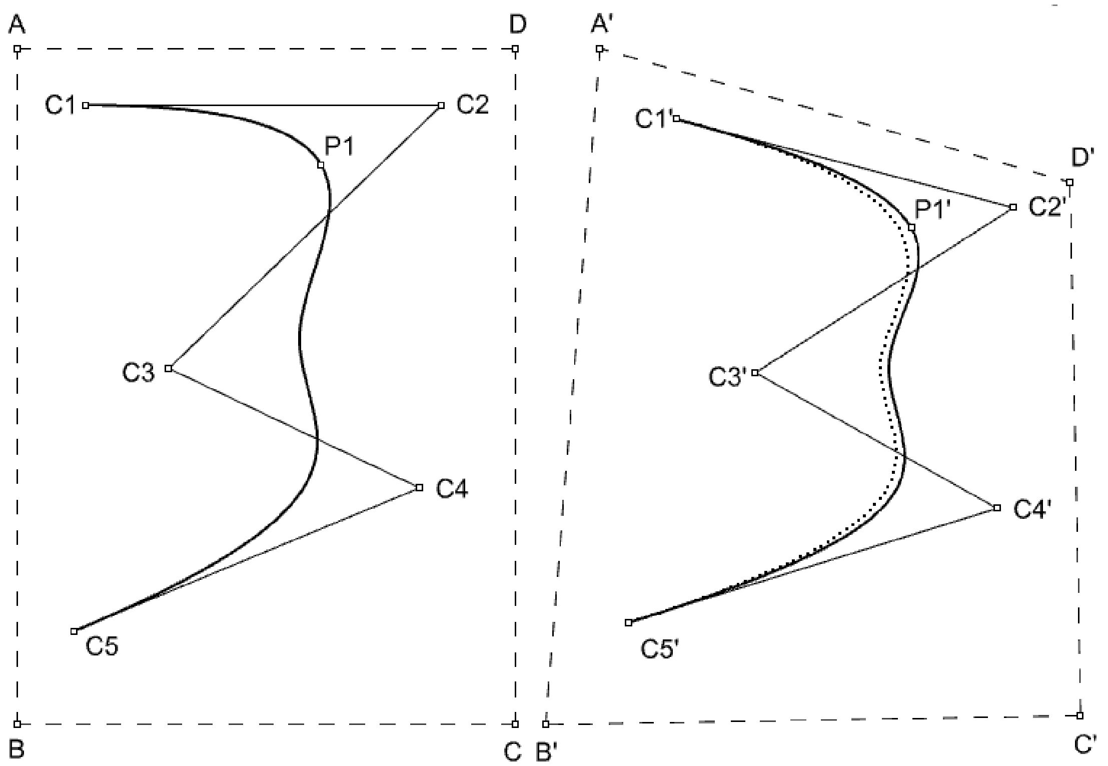


figure 15: The two homologous curves (in continuous). The dot line is a B-Spline with control points C1’, C2’, C3’, C4’ and C5’.

It should be remembered that a projective transformation transforms any quadrilateral [ABCD] into any quadrilateral [A'B'C'D’]. In the transformation illustrated in figure 15, homologous control point points could be used as control points for a new B-Spline (dotted in figure 15 on the right). But this line does not correspond to the projective transformation of the B-Spline given in the left figure. The transformed curve is the black curve in the right figure.

What line is this?

This new line is called NURBS (Non-Uniform Rational Basis Spline). The designation Non-Uniform refers to the fact that the elements of the knot vector may not be uniform (as we had already observed in B-Splines) and to a new parameter which is the definition of a weight associated with the control points whose effect is to pull or push the curve towards that point according to the higher or lower value associated with it. In a B-Spline all weights have a value of 1. In a NURBS, weights may be different from 1. If the weight of a control point is bigger than 1, the curve is drawn to the control point. If the weight of a control point is less than 1, the curve is moved away from the control point. This effect can be seen in figure 16. In this figure, a black B-Spline of degree 3 is drawn. By changing the weight of control point C5, there was an effect on the internal knots that were under the effect of that control point. Note the interesting fact that the internal knots have moved towards the control point; approached when weight was set to greater than 1 (green NURBS) and moved away when weight was set to less than 1 (blue NURBS).

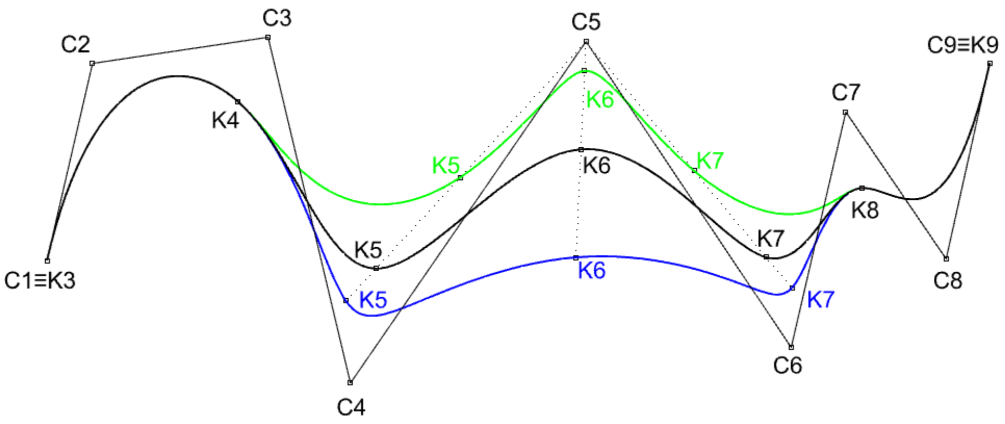


figure 16: The effect of the weight of a control point.

Since only the weight of a control point changed, the internal knots influenced by it moved in a straight line. However, if we change the weight in more than one control point, the attracting effect is combined at those points that are under the influence of more than one control point, as shown in figure 17.

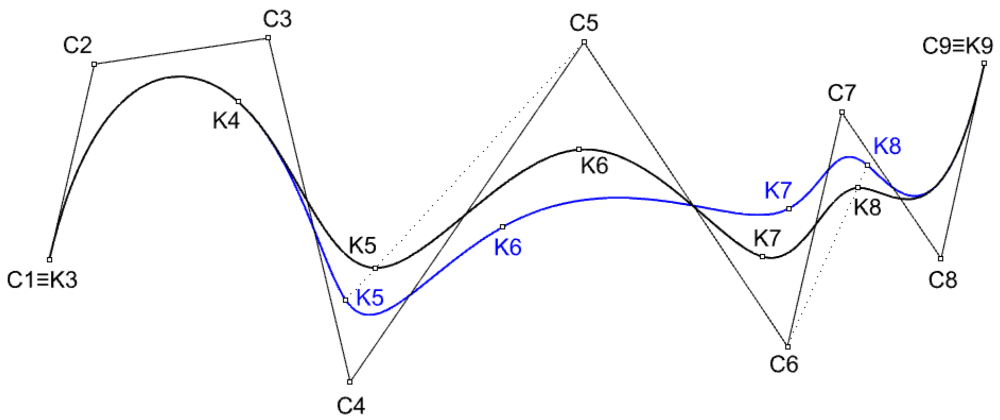


figure 17: Changing the weight of multiple control points.

A very interesting aspect about NURBS is that, considering the degree 2, and with the appropriate choice of weights, they can represent any conic line. Interestingly, therefore, the configuration illustrated in figure 8 is a particular case of a NURBS line.

3. SPATIAL LINES AND SURFACES

A spatial line is a line that doesn’t lie in a plane.

A surface can be conceptually defined as the continuous movement of a line in space, deformable or not, subject to a certain law. As has been said for lines, there is no single surface classification criterion either.

A line can result from the intersection of two surfaces. In this case it will usually be a spatial line. However, in particular cases it may be a flat line.

3.1. General notions on lines and surfaces

3.1.1. Incidence conditions

If the point P lies on the line $[d]$ and the line $[d]$ lies to the surface $[\alpha]$, then the point P belongs to the surface $[\alpha]$.

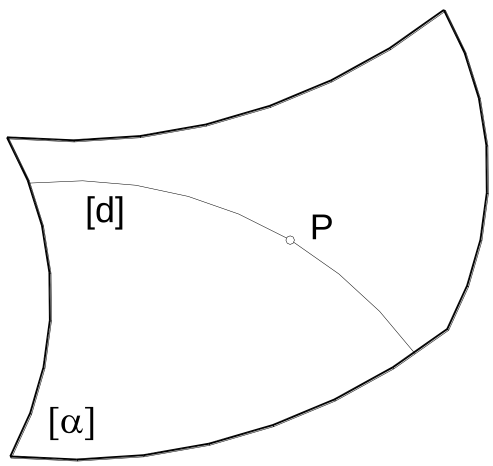


figure 18: Incidence condition of a point on a surface.

3.1.2. Tangent (straight) line

The point A lies on the line $[m]$ and the line $[m]$ lies on the surface $[\alpha]$.

The line t_A, tangent to the line $[m]$ at the point A, corresponds to the limit position of the secant line s ,when the point X tends to the point A.

If line t_A is tangent to the line $[m]$, it is also tangent to the surface $[\alpha]$.

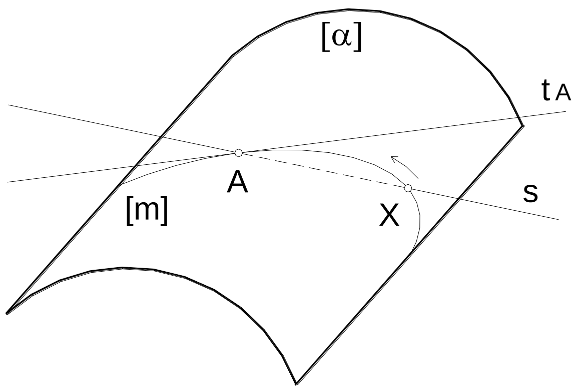


figure 19: A (straight) line tangent to a surface.

3.1.3. Curvature

A flat curved line is always contained in a plane. Other than the circumference, the CURVATURE (K) of the lines varies. The curvature of a line at a point is the inverse of the RADIUS OF CURVATURE (R) of the line at that point. And the radius of curvature of the line at one point is the radius of the OSCULATING CIRCUMFERENCE of the curve at that point. The centre of this, the point C in the figure below, can be considered as the limiting position of the intersection of two normal lines on the curve when the arc, defined by the points common to the curve and the normal, tends to zero, as shown in the following figure.

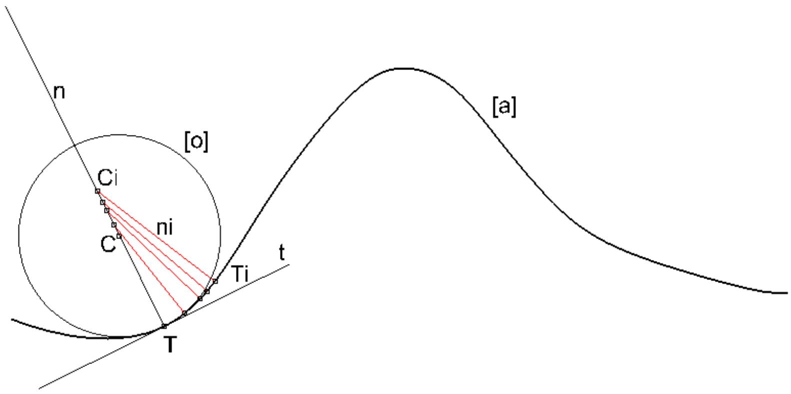


figure 20: Centre of curvature of a curve at a point T.

In the figure, it can be seen that we can assign a rectangular coordinate system associated with the straight lines t and n, and of origin T, where, as the curve is flat, it is contained in its plane. The third axis of this coordinate system is a line passing through the point T which is simultaneously perpendicular to the lines t and n which is called the BI-NORMAL line to the curve at the point T.

These concepts can be extended to SPATIAL CURVES, that is, to non-flat curves.

Two points infinitely close to a point T, on a spatial curve define an OSCULATING PLAN, which contains the osculating circumference $[c]$, the line tangent to the curve, t, and the normal line to the curve, n, at the point T. Similarly, the bi-normal line b, passes through the point T and is perpendicular to the plane of the osculating circumference.

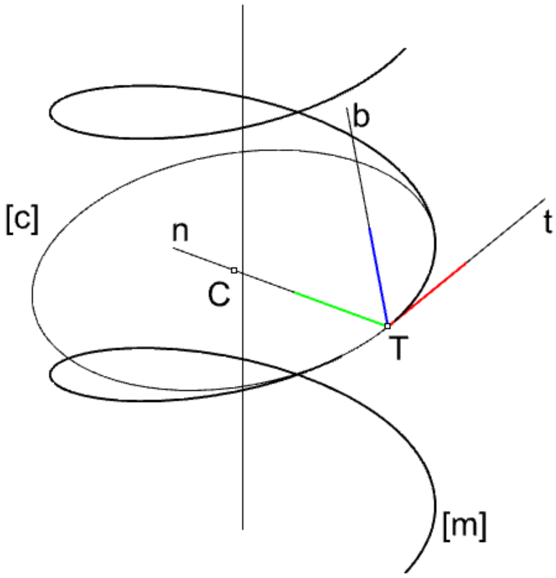


figure 21: Local frame associated with a curve at a point T.

In a spatial curve, the bi-normal rotates around the tangent as the point T moves on the curve. The higher or lower rotation rate of the bi-normal is called TORSION.

3.1.4. Continuity between curves

Between curves various types of continuity can be established. In the curves of figure 22 there is only CONTINUITY OF POSITION between the curves $[a]$ and $[b]$, that is, there is a vertex V in the transition between the curves.

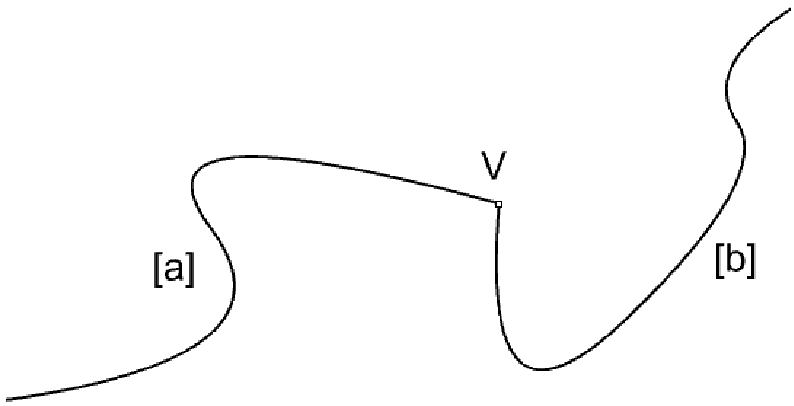


figure 22: Continuity of position.

In the curves of figure 23 there is CONTINUITY OF TANGENCY between the two curves $[a]$ and $[b]$, that is, at the transition point between the curves, both admit the same tangent line t. However, there is no continuity of curvatures, as can be seen from the graphs that illustrate the curvatures.

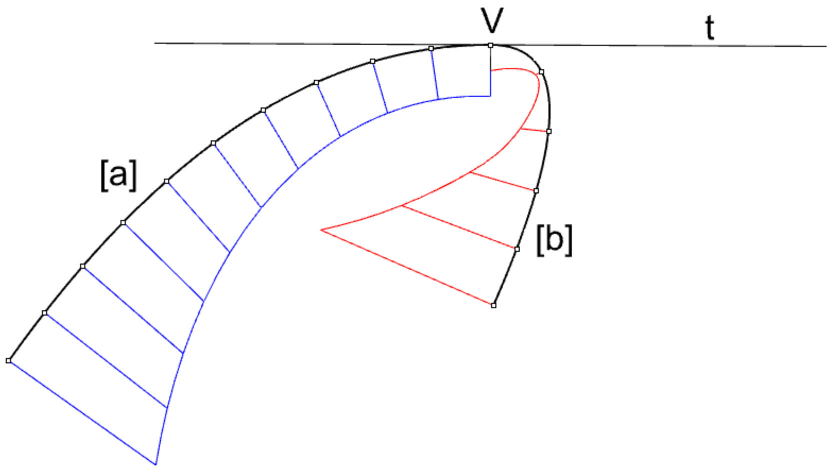


figure 23: Tangency continuity between to lines.

If, at the point V, the curvatures of both lines are the same and the lines are tangent, then the lines are said to have CURVATURE CONTINUITY.

3.1.5. Tangent plane to a surface

Let two lines, $[a]$ and $[b]$ belonging to the surface $[\alpha]$, be incident at the point P.

Let t_a and t_b be the lines tangent to the lines $[a]$ and $[b]$, respectively, at the point P.

The plane ε, defined by the lines t_a and t_a, is the plane tangent to the surface $[\alpha]$ at the point P.

From the tangent plane to a surface it is said to be the OSCULATING PLANE.

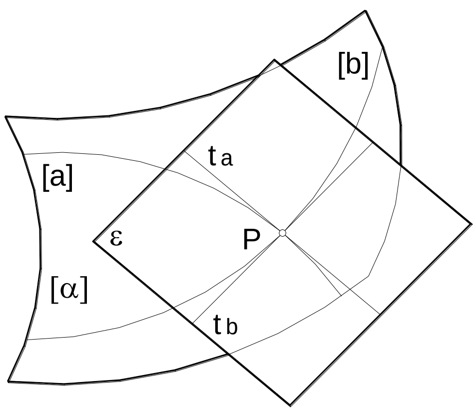


figure 24: Tangent plane to a surface at a point.

3.1.6. Normal (straight) line and normal plane

Let the plane ε be tangent to the surface $[\alpha]$ at the point P.

Be n a line perpendicular to the plane ε at the point P.

The line n is said to be the NORMAL LINE to the surface $[\alpha]$ at the point P.

From a plane containing the line n it is said to be a NORMAL PLANE to the surface $[\alpha]$ at the point P.

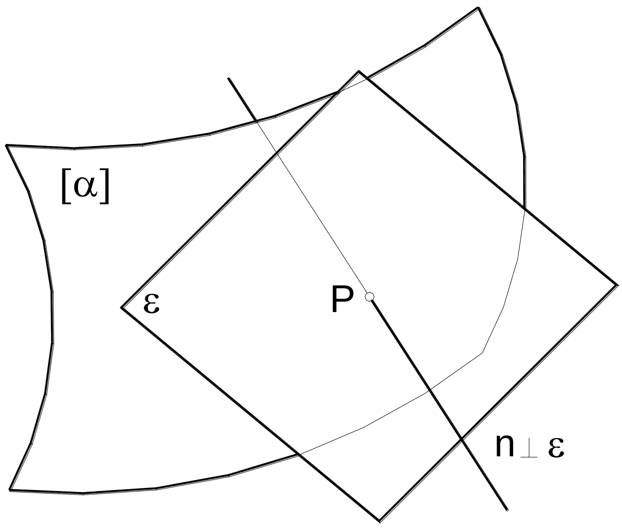


figure 25: Normal line to a surface at a point P.

3.1.7. Surface curvature

Let n be a normal line to the surface $[\alpha]$ at the point P.

Let π and β be the normal planes to the surface $[\alpha]$ at the point P.

Let $[c]$ (result of the intersection of the plane π with the surface $[\alpha]$) be the line of maximum curvature of the surface $[\alpha]$ at the point P.

Let $[d]$ (result of the intersection of the plane β with the surface $[\alpha]$) be the line of minimum curvature of the surface $[\alpha]$ at the point P.

The planes containing the maximum and minimum curvature lines at a point are found to be perpendicular to each other.

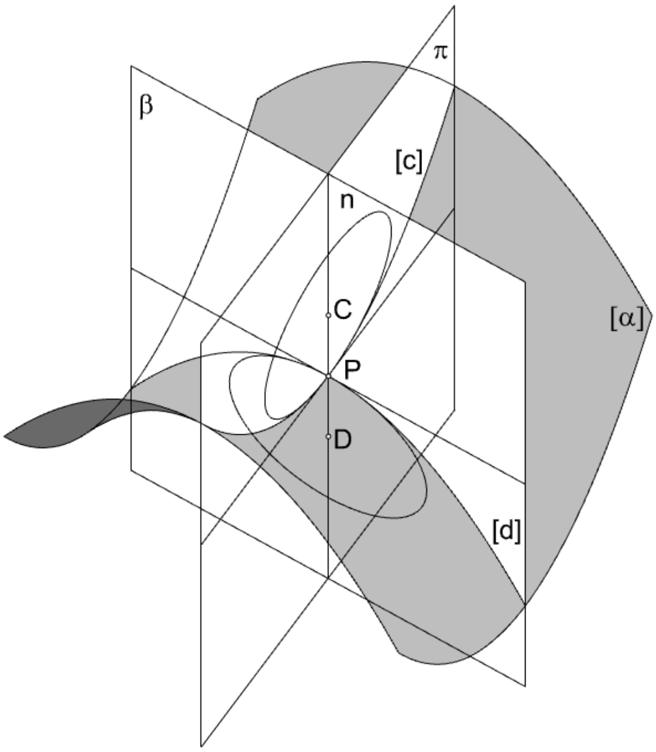


figure 26: Principal normal planes and principal normal sections to a surface at a point P.

If the tangent plane to the surface $[\alpha]$ at the point P divides the surface into four regions, two "up" from the plane and two "down", then the surface has DOUBLE CURVATURE OF OPPOSITE SENSES at the point P. The surface is ANTICLASTIC at P.

If the tangent plane to the surface $[\alpha]$ at the point P only contains P in its vicinity, then the surface has DOUBLE CURVATURE WITH THE SAME SENSE at the point P. The surface is SYNCLASTIC.

If the tangent plane to the surface $[\alpha]$ at the point P has in common with $[\alpha]$ only one passing line through P, then the surface has SIMPLE CURVATURE at the point P.

3.1.7.1. Mean curvature

The MEAN CURVATURE (K_m) of the surface $[\alpha]$ at the point P is the average of the maximum and minimum curvatures.

3.1.7.2. Gaussian curvature

GAUSSIAN CURVATURE (K_g) of the surface $[\alpha]$ at the point P is the product of the maximum and minimum curvature.

A developable surface has zero Gaussian curvature at all points.

3.1.8. Surfaces intersection

If two surfaces $[\alpha]$ and $[\beta]$ intersect on a line $[i]$, then there is at least one surface $[\pi]$ that intersects the surface $[\alpha]$ on a line $[a]$, intersects the surface $[\beta]$ on a line $[b]$, such that the line $[a]$ intersects the line $[b]$ at a point I on the line $[i]$.

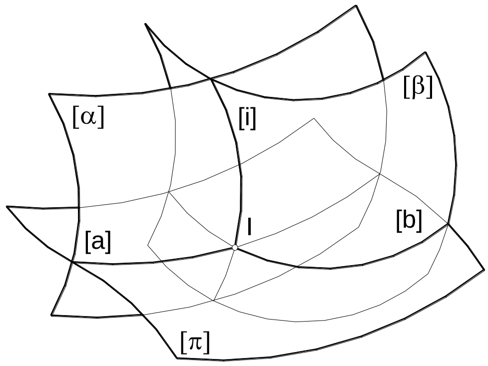
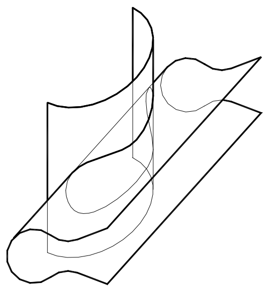


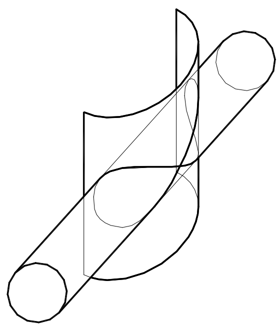
figure 27: Intersection between surfaces.

According to the intersection line, intersections can be classified in three ways. If the intersection line is single and closed there is a ONE PART INTERSECTION also called a PULLOUT INTERSECTION.



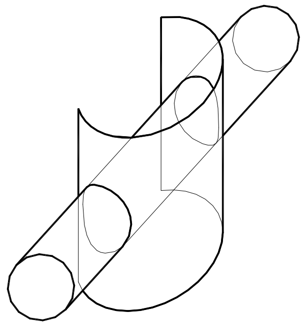
One part Intersection.

If the intersection line has a double point, the it is a curve with a SINGULAR POINT.



Singular point intersection.

It there are two intersection lines then there is a TWO PARTS INTERSECTION[6] also called PENETRATION.



Two parts intersection.

3.1.9. Line tangent to the intersection line

Let $[i]$ be the line of intersection between surfaces $[\alpha]$ and $[\beta]$.

Be P a point of the line $[i]$, therefore a common point between $[\alpha]$ e $[\beta]$.

Let the plane δ be tangent to the surface $[\alpha]$ at the point P.

Let the plane π be tangent to the surface $[\beta]$ at the point P.

The line t, intersecting of the planes δ and π, is the line tangent to the line $[i]$ at the point P.

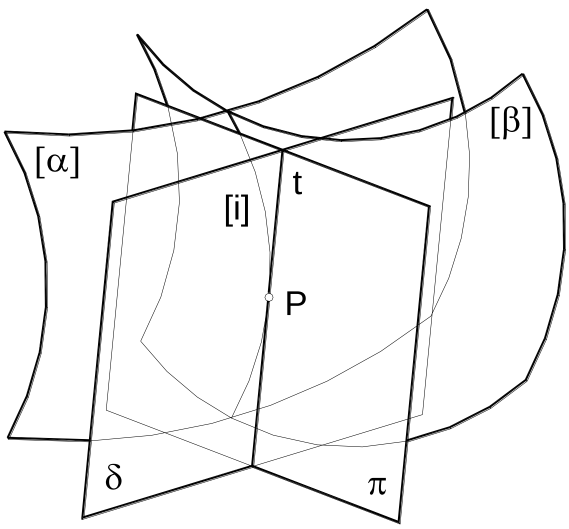


figure 28: Tangent line as the result of the intersection of tangent planes.

[6] See https://en.wikipedia.org/wiki/Intersection_curve .

3.1.10. Tangency between surfaces

If two surfaces $[\alpha]$ and $[\beta]$ admit the same tangent plane s π at all points P of the line common to both, then the two surfaces are said to tangent along the line $[c]$. Meeting this condition ensures the "visually smooth transition" between surfaces, which is called G1 continuity. However, there may be discontinuity at the curvature level. If there is continuity between surfaces at curvature level, it is said that there is continuity G2. A common test for assessing the type of continuity between surfaces is ZEBRA analysis.

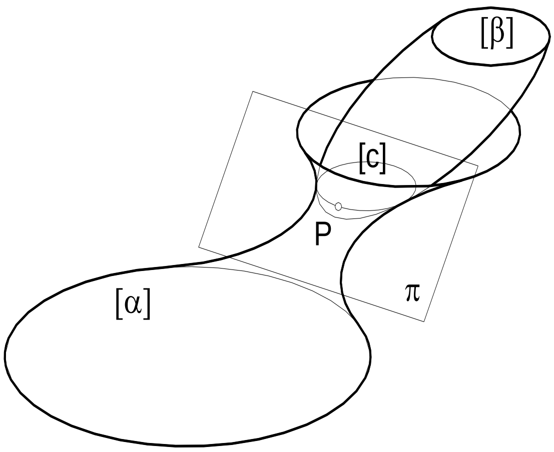


figure 29: Two surfaces tangent along a common line.

If two surfaces $[\alpha]$ and $[\beta]$ are tangent along a line $[i]$, then there is at least one surface $[\pi]$ that intersects the surfaces $[\alpha]$ and $[\beta]$ along the lines $[b]$ and $[a]$, respectively, such that the lines $[b]$ and $[a]$ are tangent to each other at a point I on the line $[i]$.

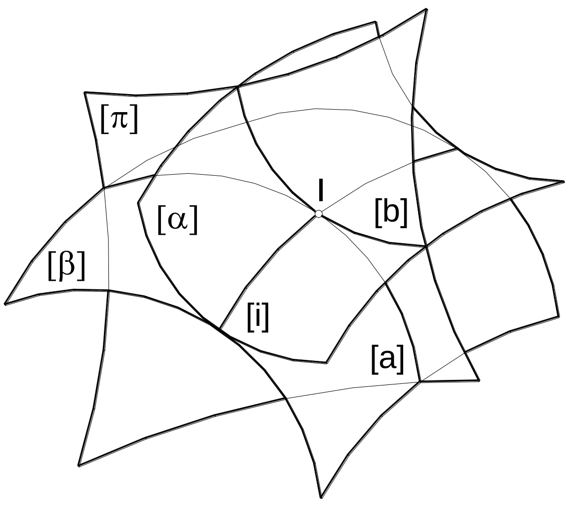


figure 30: Tangent surfaces and tangent lines.

If two surfaces $[\alpha]$ and $[\beta]$ are tangent along a line $[i]$ and are both tangent to a surface $[\pi]$ along the lines $[a]$ and $[b]$, respectively, such that $[a]$ and $[b]$ intersect at point I on the line $[i]$, then the two lines $[a]$ and $[b]$ are tangent to each other at the point I .

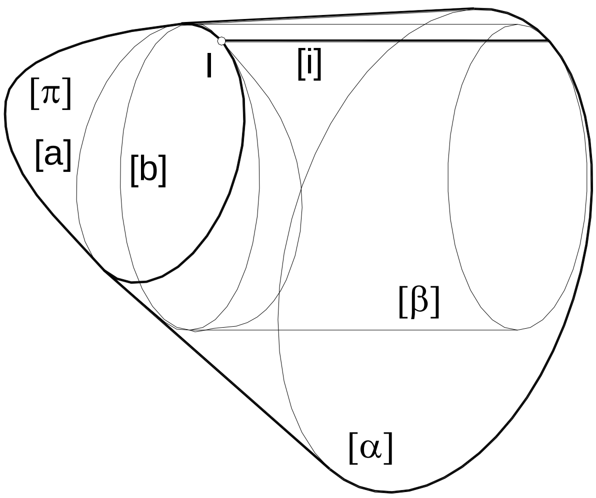


figure 31: Tangent surfaces and tangent lines.

The same discussion of continuity about lines can be transposed to surfaces.

3.1.11. Silhouette

The silhouette of a surface $[\alpha]$ for an "observer" O (projection centre) is the line of tangency $[c]$ between the surface $[\alpha]$ and a conical surface $[\pi]$ with vertex O.

If the observer is at infinity, then $[\pi]$ is a cylindrical surface.

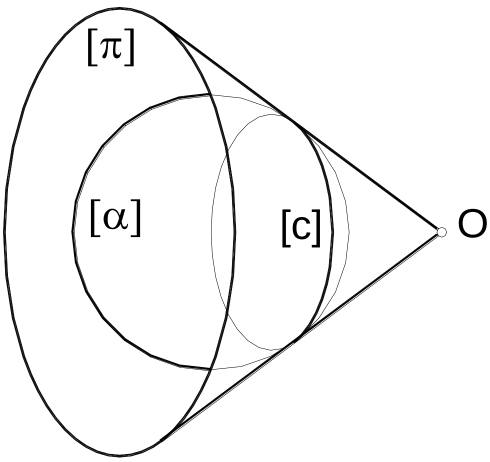


figure 32: Silhouette of a surface seen from a viewpoint O.

3.1.12. Distinction between surface and solid

Uma superfície é a entidade que delimita o volume do sólido.

3.2. Surface classification by type of generatrix

SURFACE CLASSIFICATION BY GENERATRIX TYPE			examples
		POLYEDRIC SURFACES	regular, semi-regular, irregular polyhedral meshes
RULED (generated by straight lines)	DEVELOPABLE	PLANAR SURFACE	llane
		defined by 1 POINT and 1 DIRECTRIX	conical; cylindrical; prismatic; pyramidal
		defined by 2 DIRECTRICES	Convoluted surfaces; surfaces of constant slope
		TANGENTIAL SURFACES	tangential helicoid
		other surfaces	
	NON DEVELOPABLE	defined by 3 DIRECTRICES	hyperbolic paraboloid; hyperboloid of revolution; cylindroid; conoid; ruled helicoids; skewed arc surfaces
		Other surfaces	single face ruled surface
CURVED (not generated by straight lines)		REVOLUTION SURFACES[7]	spherical; toric; ellipsoidal; others
		Others	serpentine; minimal surfaces; NURBS [8]

Tabela 1: Classification of surfaces.

3.2.1. Polyedric surfaces

The relationship between the number of edges (A), vertices (V) and faces (F) of any polyhedron topologically equivalent to a sphere is given by Euler's formula:

$$V + F = A + 2 \qquad (6)$$

A polyhedron whose vertices are the centers of the faces of another polyhedron is called DUAL of that one.

3.2.1.1. Regular polyhedra

All faces are regular polygons of only one type, all vertices belong to a spherical surface, they are the "Platonic Solids".

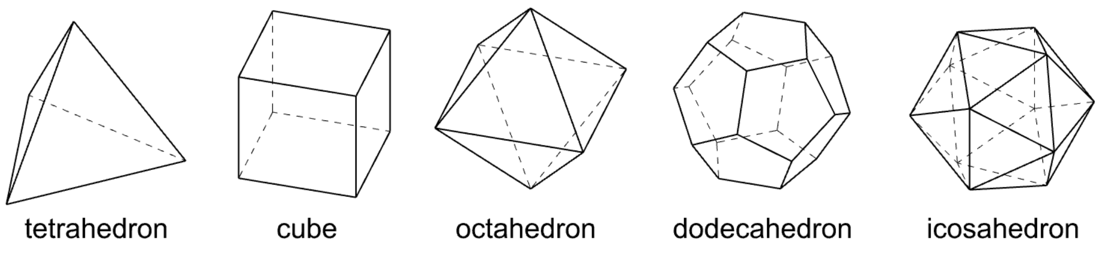


figure 33: The five regular polyhedra

[7] Note that there are surfaces of revolution that are developable (conical and cylindrical). Note that the surface of the hyperboloid of revolution can also be defined by three straight generatrices.
[8] Note that NURBS surfaces can be used as a representation of many other surfaces used in the table.

3.2.1.2. Semi-regular polyhedra

All faces are regular polygons of two or more types with constant edge length; all vertices belong to a spherical surface. They are also referred to as “Archimedes Solids”. All edges and vertices are congruent and can be obtained from regular polyhedra by some process of geometric transformation. This category may also include regular prisms and regular anti-prisms although this is not normally common.

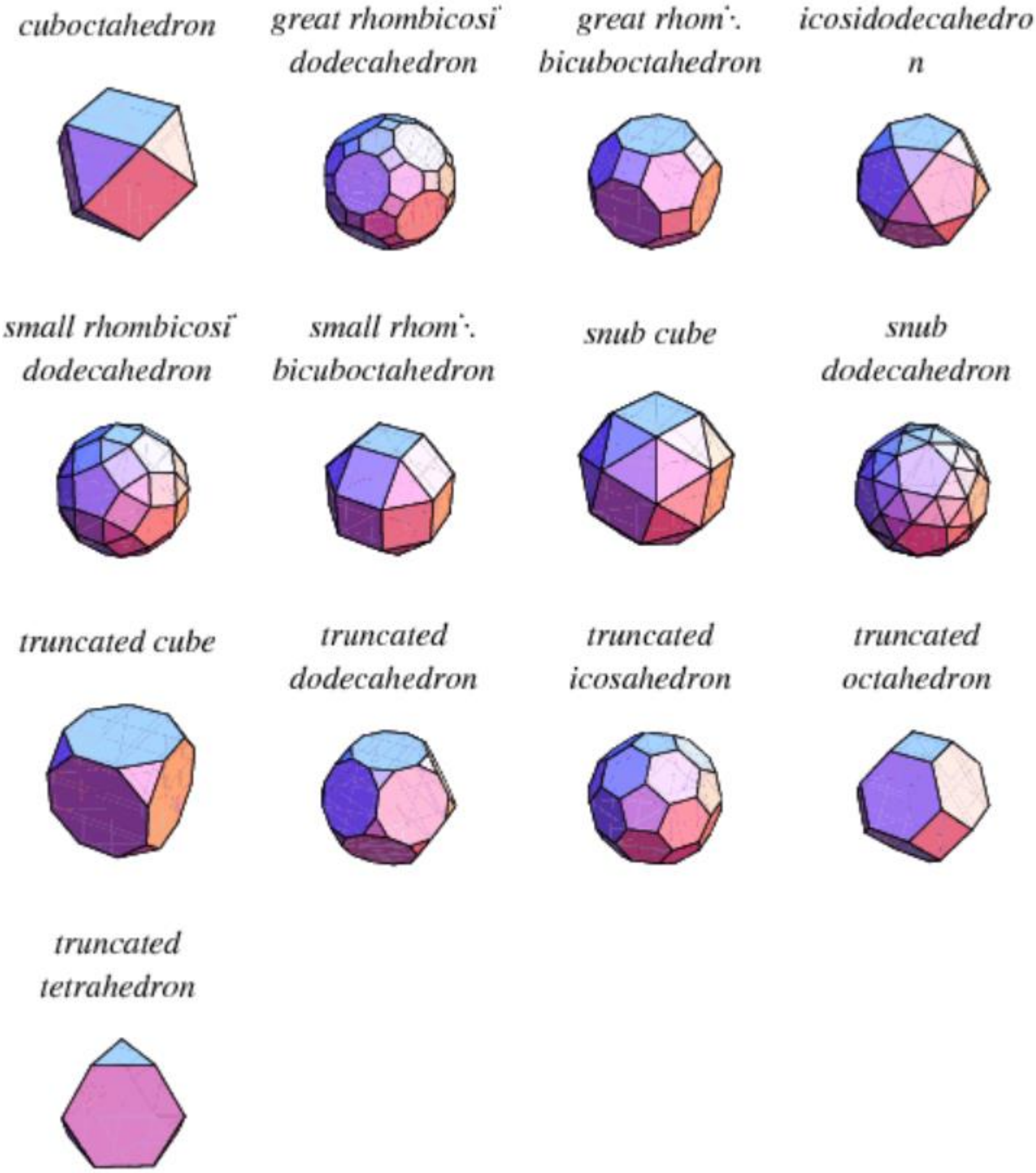


figure 34: The thirteen Archimedean Solids (in http://mathworld.wolfram.com/ArchimedeanSolid.html)

3.2.1.3. Irregular polyhedra

Faces are polygons of various types. Vertices may or may not belong to a spherical surface. The length of the edges is not constant. Examples of irregular polyhedra are pyramids, bipyramids, trunks of pyramid, prisms, trunks of prism. A bipyramid is a solid generated by the “sum” of a pyramid with its symmetrical relative to the base plane. Other types of irregular polyhedra are antiprisms, antipyramids, trunk of antiprisms and antiprismoids.

An antiprismoid has two opposite faces which are polygons with different number of sides.

In an antipyramid one of the polygons is replaced with an edge.

An antiprism has two opposite faces which are polygons with the same number of sides.

A trunk of antiprism has two opposite faces which are polygons with the same number of sides but in non-parallel planes.

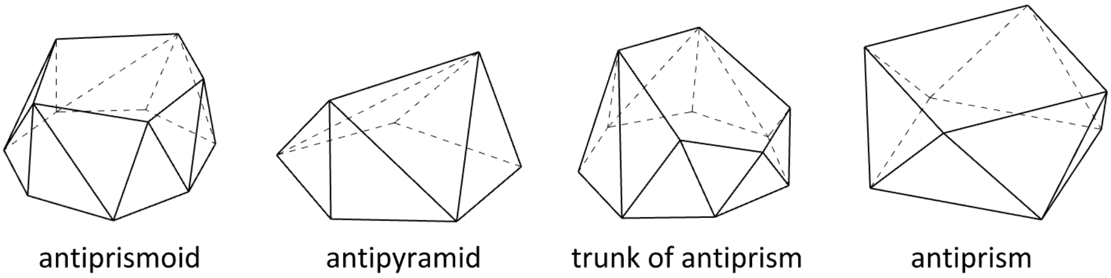


figure 35: Examples of irregular polyhedra.

Another interesting type of polyhedron are Johnson's solids. These are polyhedra in which all faces are regular of more than one type, but are not regular, semi-regular, regular prisms, or regular anti-prisms. There are 92 in all.

3.2.2. Surfaces of revolution

A revolution surface is generated by the rotation of a GENERATRIX around an AXIS.

A PARALLEL is an intersection produced at the surface by a plane perpendicular to the axis.

A MERIDIAN is an intersection produced on the surface by a plane complanar with the axis.

If a parallel is the largest in its vicinity, it is called ECUATOR.

If a parallel is the smallest in its vicinity, it is called a “BOTTLENECK” CIRCLE.

If the surface admits tangent planes perpendicular to the axis at points which it has in common with that axis, then these points are called POLES.

If the surface admits tangent planes perpendicular to the axis along parallels, these are called POLAR CIRCLES.

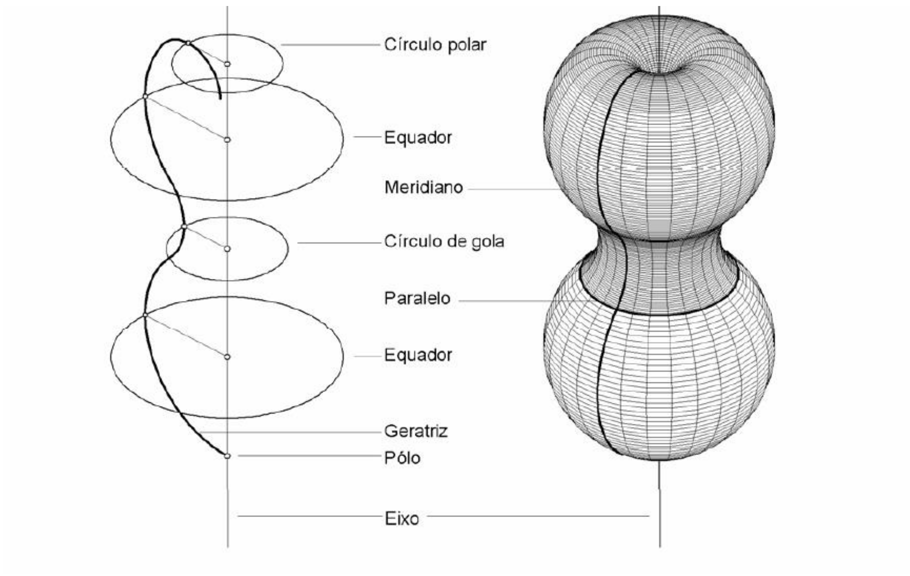


figure 36: Surface of revolution.

3.2.2.1. Spherical surface

The spherical surface may be conceptually designed as the result of rotating a circumference about a diameter.

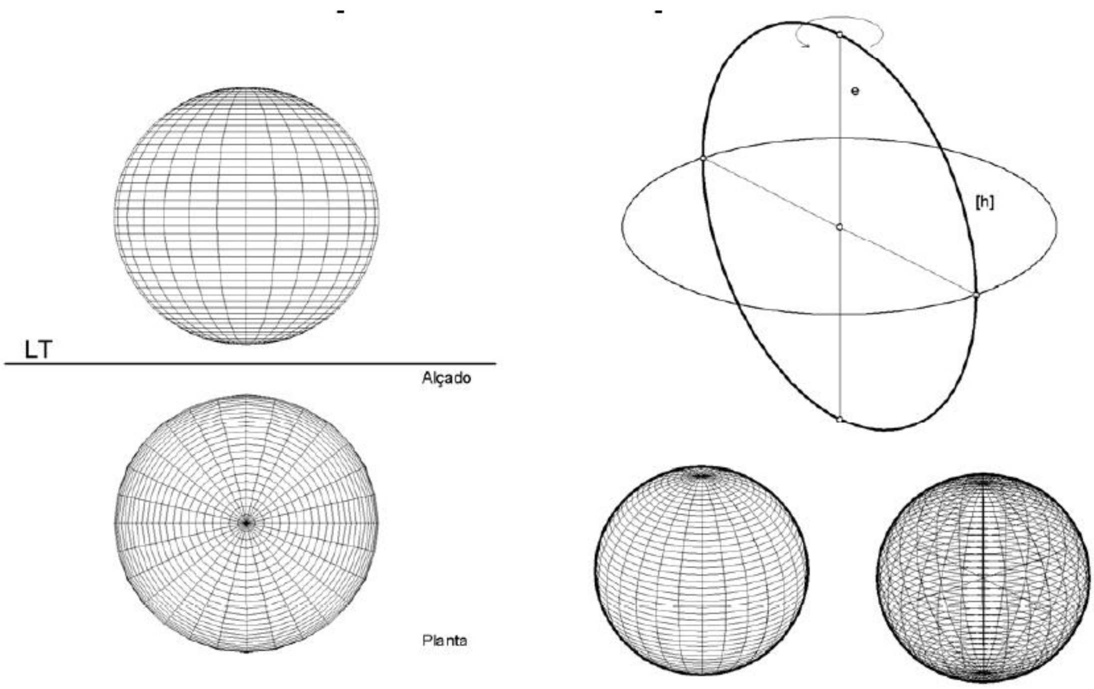


figure 37: Generating a sphere by rotating a circumference around a diameter.

3.2.2.2. Spheroid

The spheroid, or revolution ellipsoid, is a surface that is generated by rotating an ellipse around one of its major axes. It is said to be ELONGATED if the rotation is made about the major axis, and it is FLATENED if the rotation is made about the minor axis.

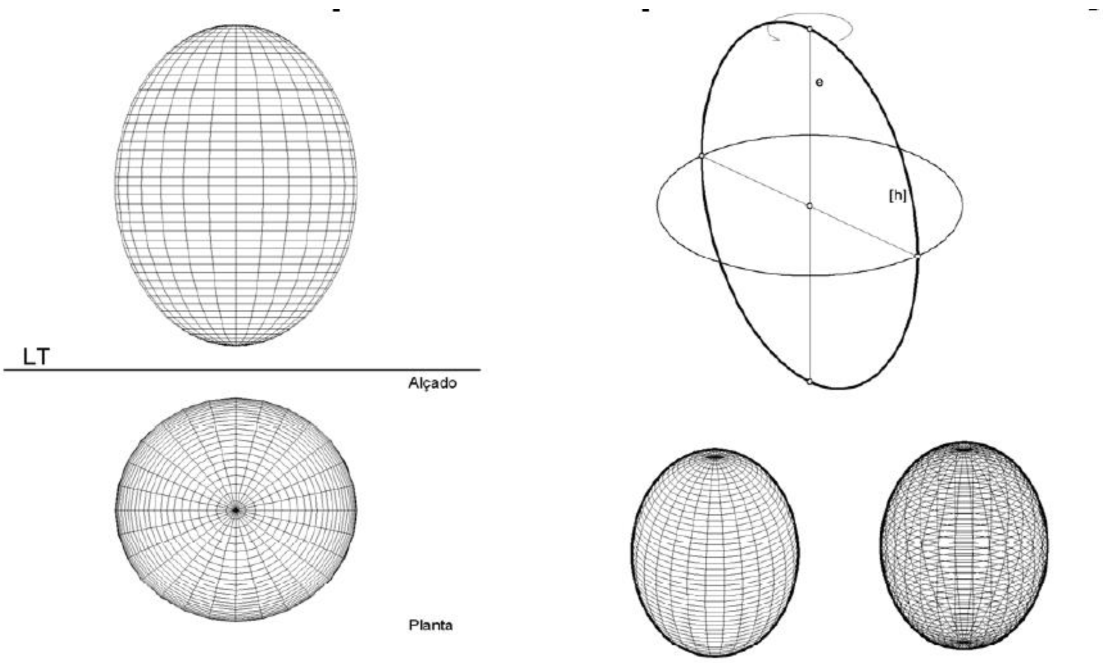
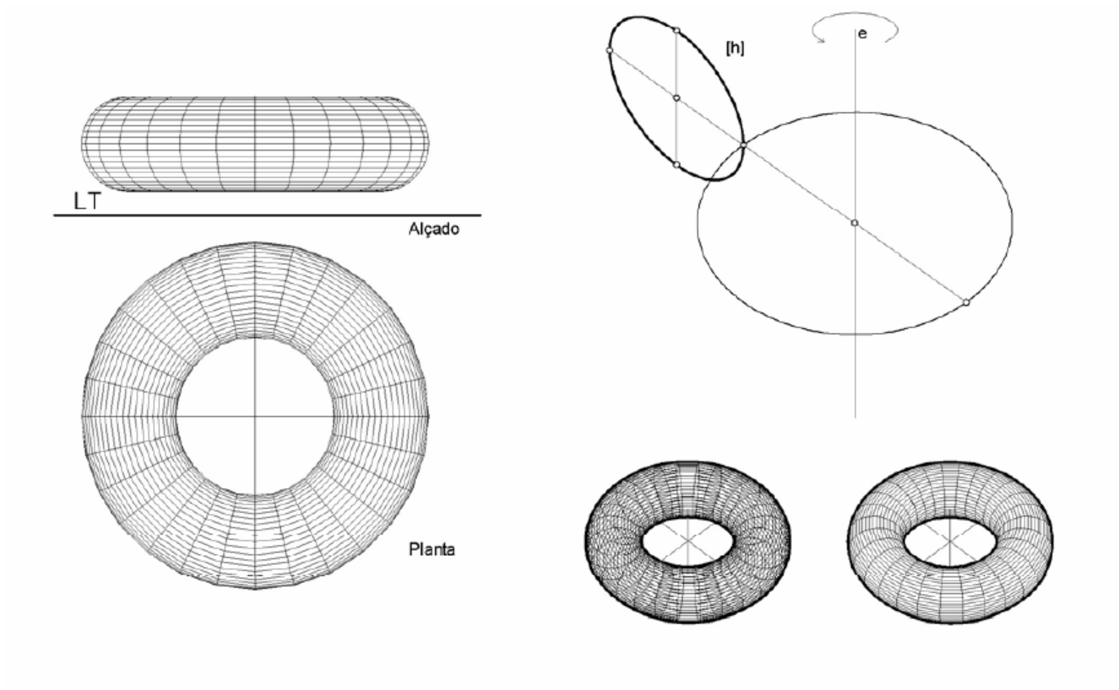


figure 38: Generation of the spheroid by rotating an ellipse about one of its major axis.

3.2.2.3. Toric surface

The toric surface is generated by rotating circumference about an axis contained in its plane. The axis is considered not to intersect the circumference.



GERAÇÃO DO TORO POR ROTAÇÃO DE UMA CIRCUNFERÊNCIA EM TORNO DE UM EIXO COMPLANAR

figure 39: Seneration of a toric surface by rotating a circumference around a copalanar axis.

3.2.2.4. Revolution hyperboloid of one sheet

The revolution hyperboloid of one sheet, which will later be seen to be also a ruled surface, is generated by the rotation of a hyperbola about its conjugated or non-transverse axis.

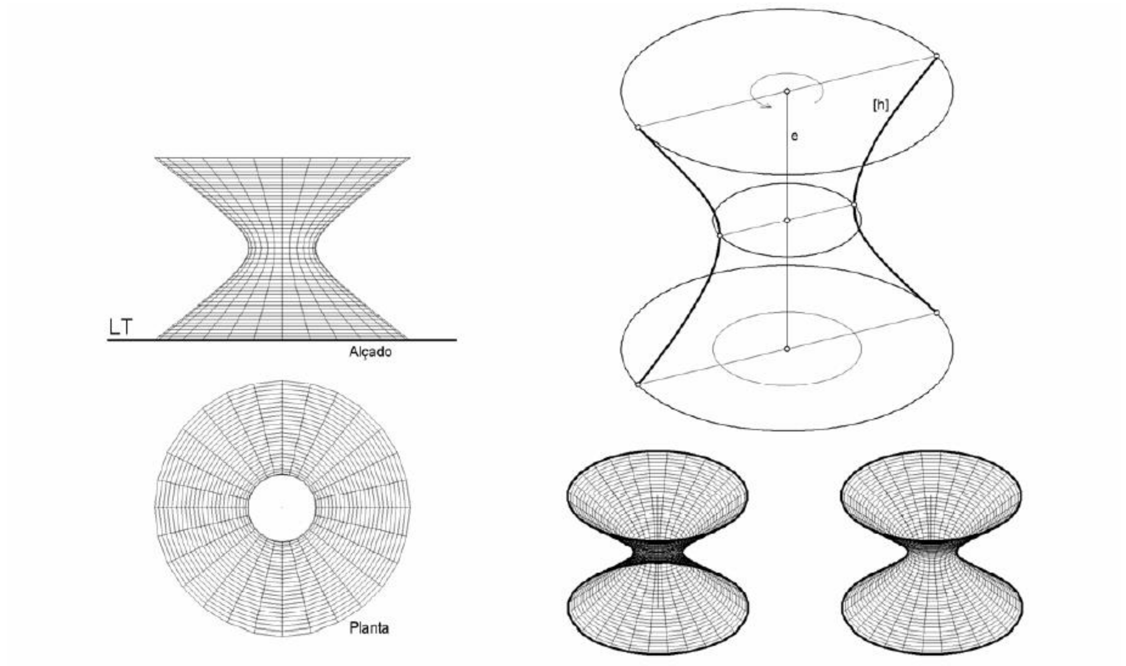


figure 40: Generating the one sheet hyperbolid of revolution by rotating an hyperbola.

3.2.2.5. Revolution hyperboloid of two sheets

The two-sheet revolution hyperboloid is the surface generated by the rotation of the hyperbola around its transverse axis.

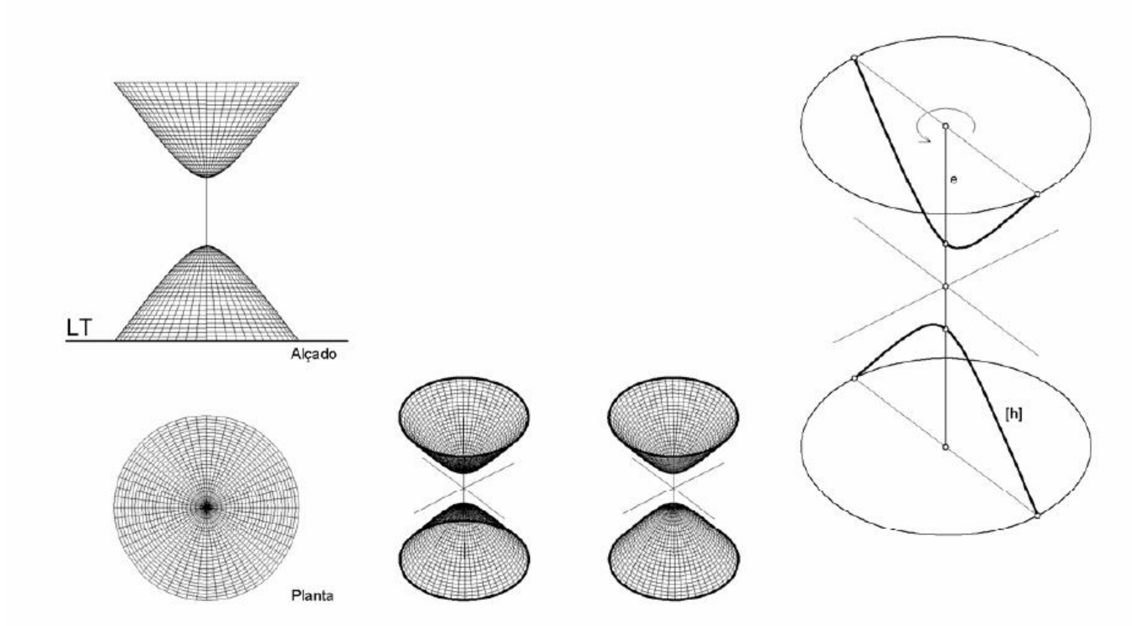


figure 41: Generating the two sheet hyperbolid of revolution by rotating an hyperbola.

3.2.2.6. Revolution paraboloid

The paraboloid of revolution is the surface generated by the rotation of the parabola about its axis.

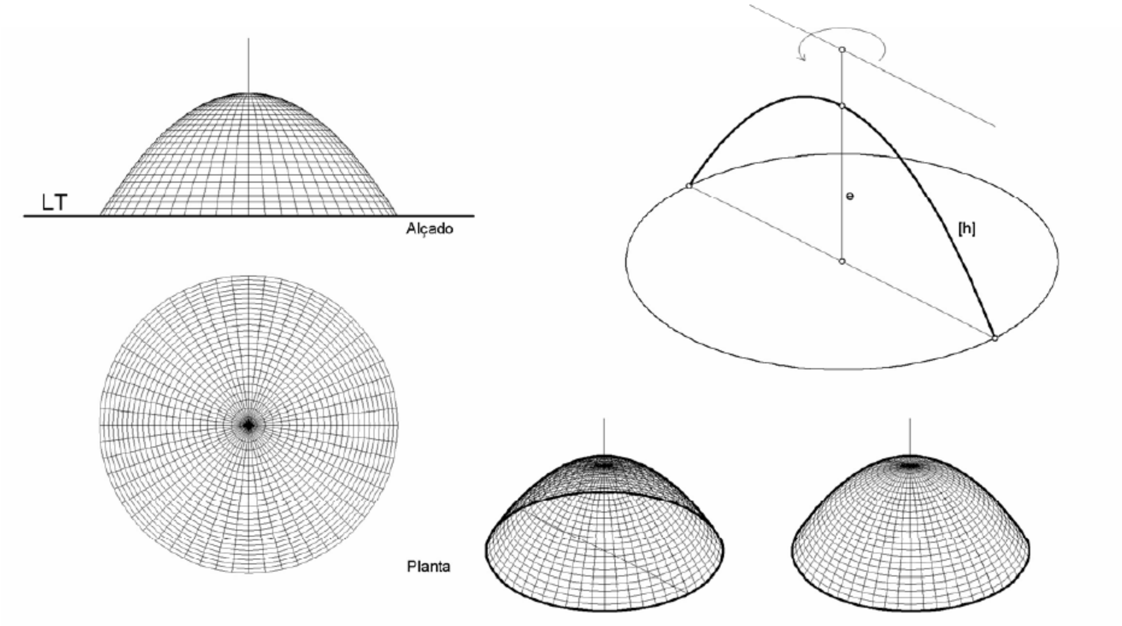


figure 42: Paraboloid of revolution by rotation a parabola around its axis.

3.2.3. Developable surfaces

For a surface to be developable it must be ruled. But this condition alone does not imply that the surface is developable. In addition to being ruled, it must also happen that each pair of generatrices infinitely close to each other are concurrent, that is, coplanar. From the statement it follows that a developable surface only admits one tangent plane for each generatrix. To development of a surface is the “unrolling” of the surface until it coincides with one of the tangent planes. In this operation the surface does not “stretch” or “shrink”, “tear” or “fold”. In this operation the lengths and angles are preserved. Solving concrete problems obviously depends on the surface present. Thus, different methods will be used to develop conical or cylindrical surfaces of revolution, conical or cylindrical oblique, convoluted, tangential, etc.

As noted above, a developable surface has a Gaussian curvature at all points equal to zero.

3.2.3.1. Conical, cylindrical, piramidal and prismatic surfaces

The most common developable surfaces are conical, cylindrical, pyramidal and prismatic surfaces.

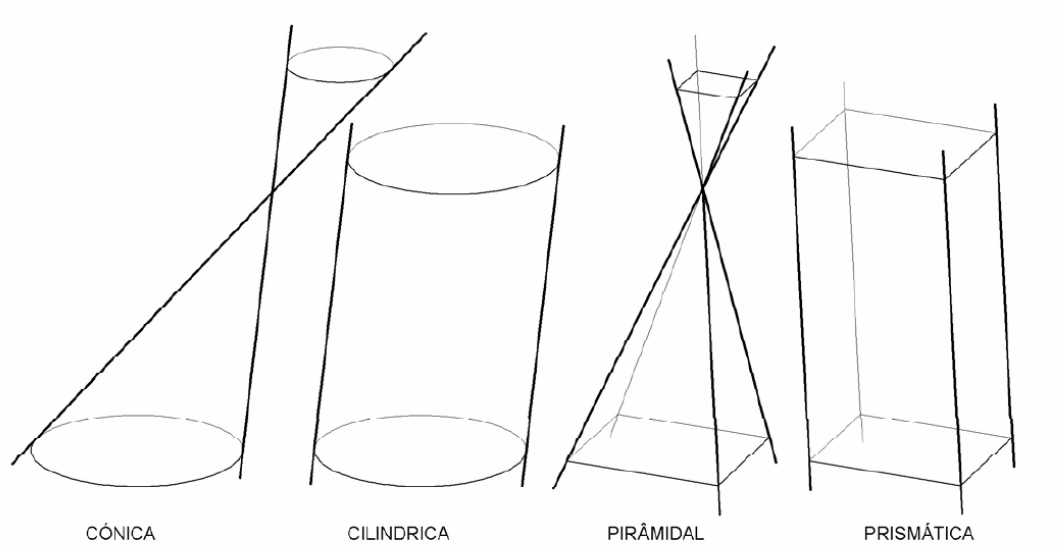


figure 43: Developable surfaces passing through a point (that can be at infinity).

3.2.3.2. Convoluted surface and tangential surface

The convolute is defined by the geometrical locus of the lines that result from the tangency points of a pair of coplanar lines tangent to two given curved lines, which move in space while maintaining those conditions (tangency and coplanarity).

Tangential surfaces are defined by the locus of the tangent lines to a spatial line.

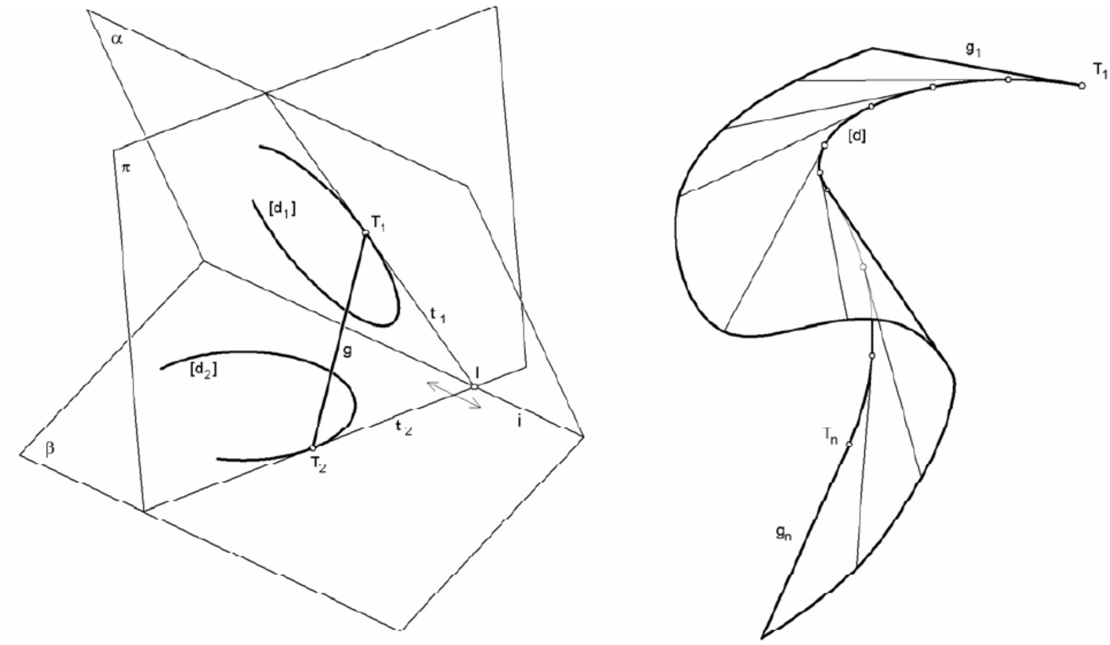


figure 44: Convuluted surface (left) and tangential surface (right).

3.2.3.3. Tangential helicoid

The tangential helicoid is a particular case of a tangential surface. In this case the directrix is a cylindrical helix.

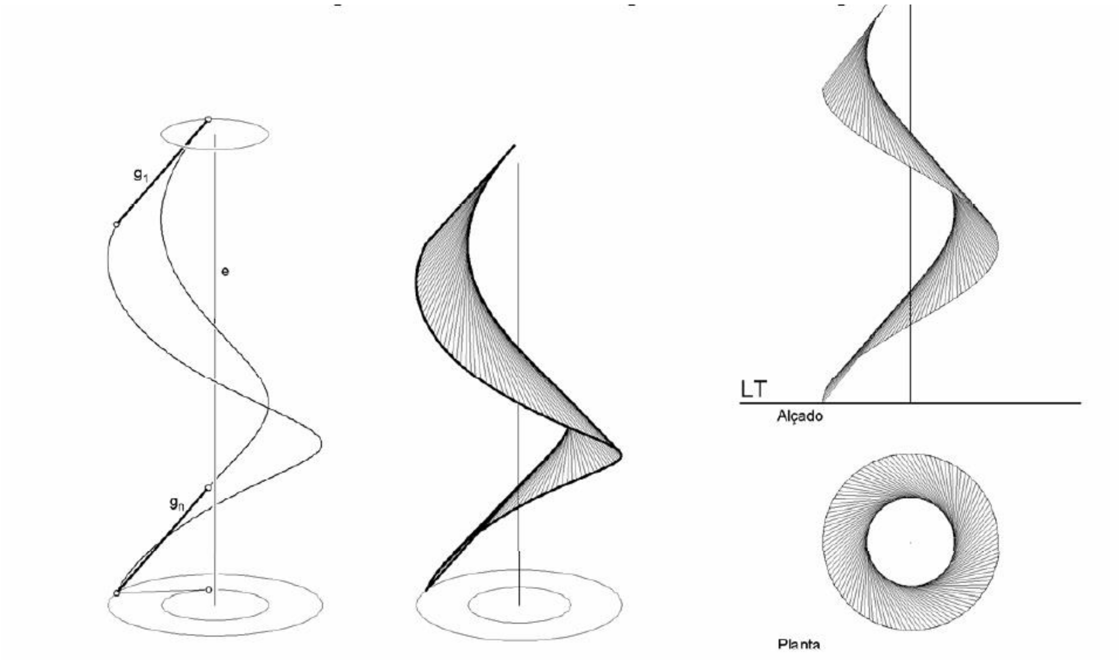


figure 45: Tangential helicoid.

3.2.3.4. Unrolling (graphical method)

Developing a surface consists of unrolling it to a plane. One approach to this process is to define the surface by triangulation and then adjust these triangles in the plane. This process turns out to be valid for "developing" non-developable surfaces as all surfaces can be approximated by triangulation.

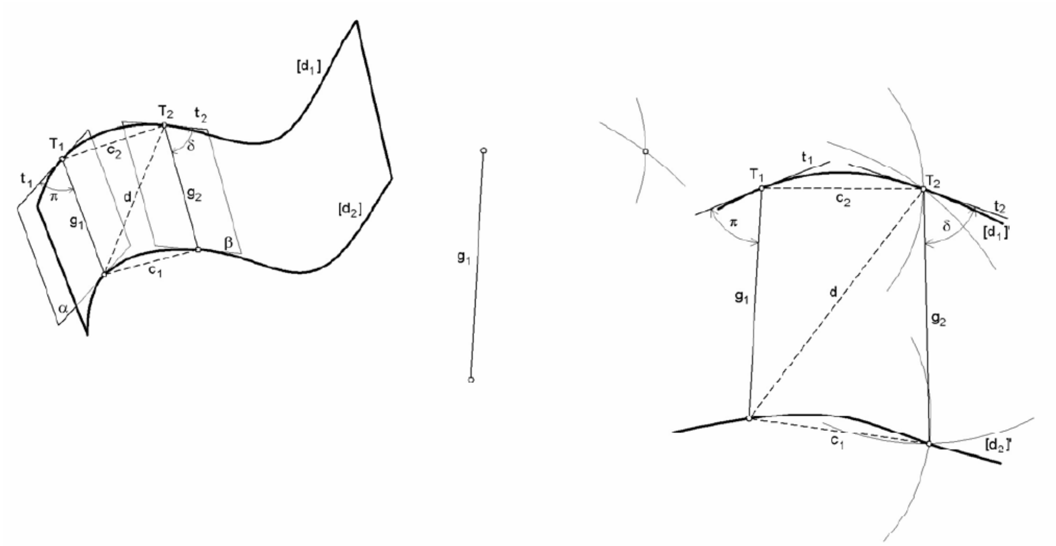


figure 46: Developing a surface (graphical method).

Some 3D modelling tools implement surface unrolling functions, such as Rhinoceros software. However, it turns out that the results are not correct when one tries to generalize the operation to non-developable surfaces.

3.2.3.5. Development of the surfaces of the revolution cone and revolution cylinder

However, there are figures whose development is quite simple. It is the unrolling of the surface of the revolution cylinder, which results in a rectangle, and the unrolling of the surface of the cone of revolution that results in a circular sector.

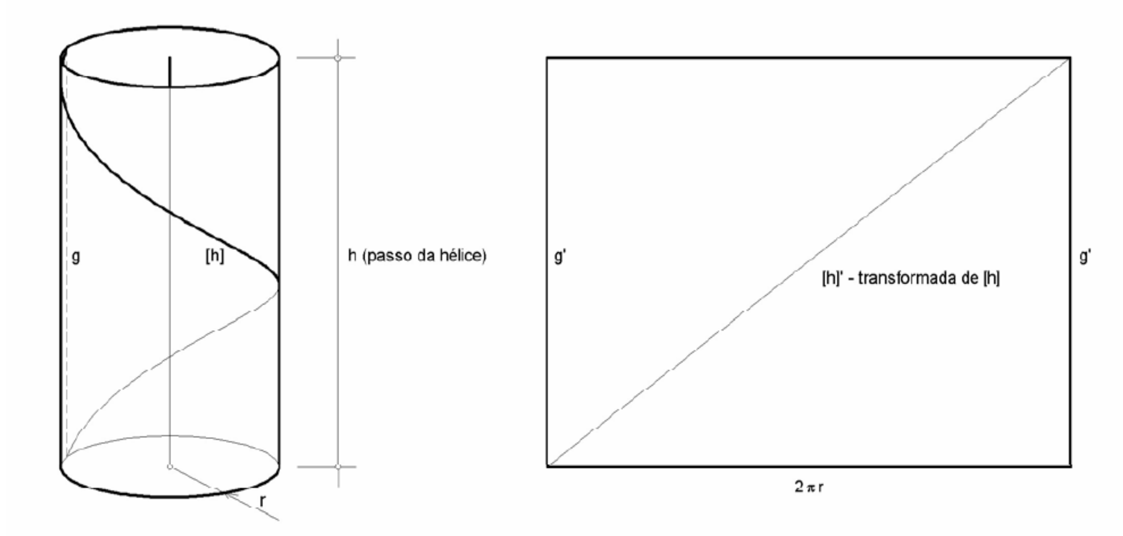


figure 47: Developing the surface of a revolution cylinder considering the transformed line of the helix.

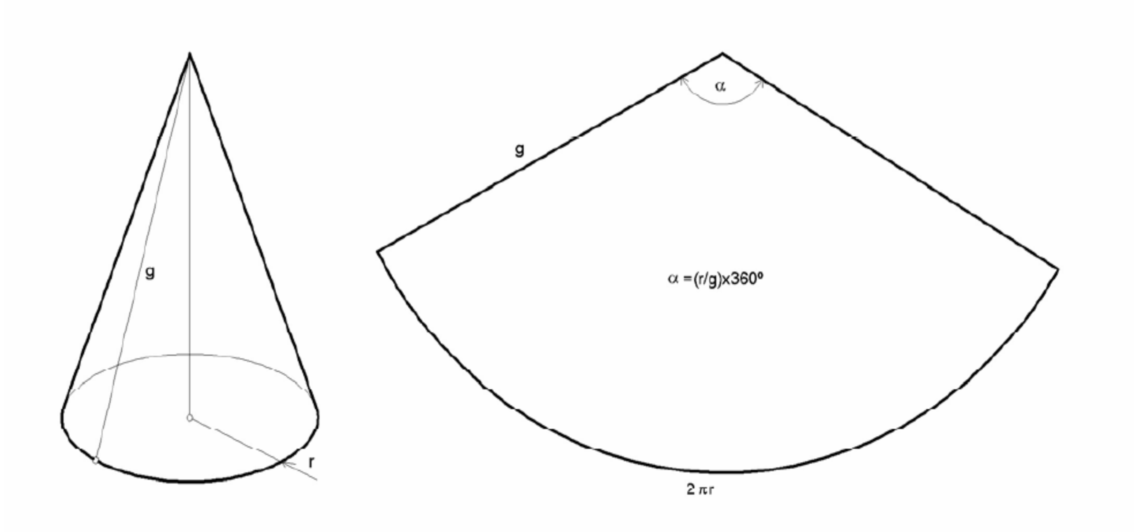


figure 48: Developing the surface of a revolution cone.

3.2.3.6. Developing the surfaces of the oblique cylinder and cone

The oblique cylinder can be unrolled by approximating its surface to that of a prism, and the oblique cone can be flattened by approximating its surface to that of a pyramid. The result is better the more refined the approximation.

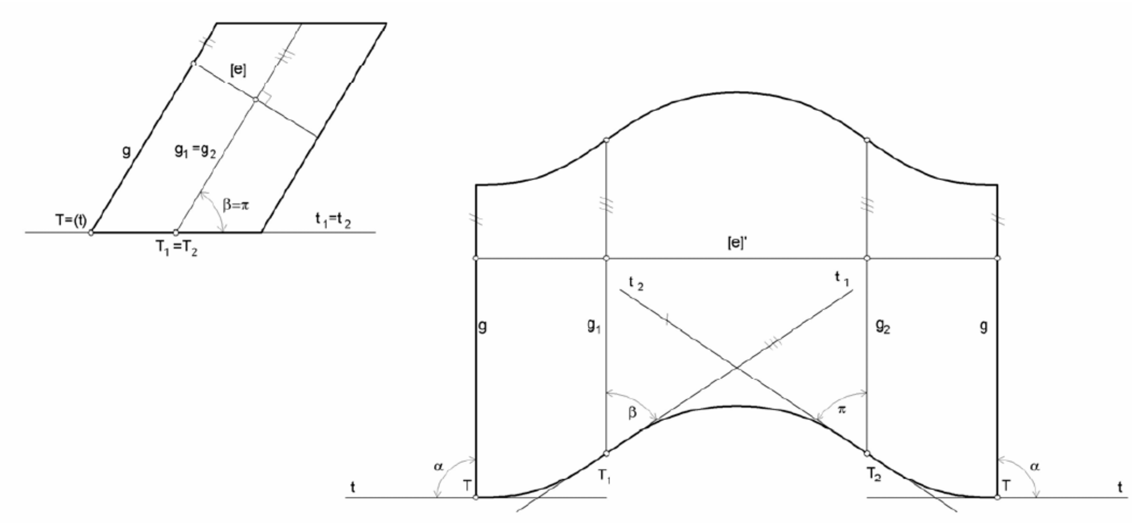


figure 49: Unrolling of the surface of the oblique cylinder.

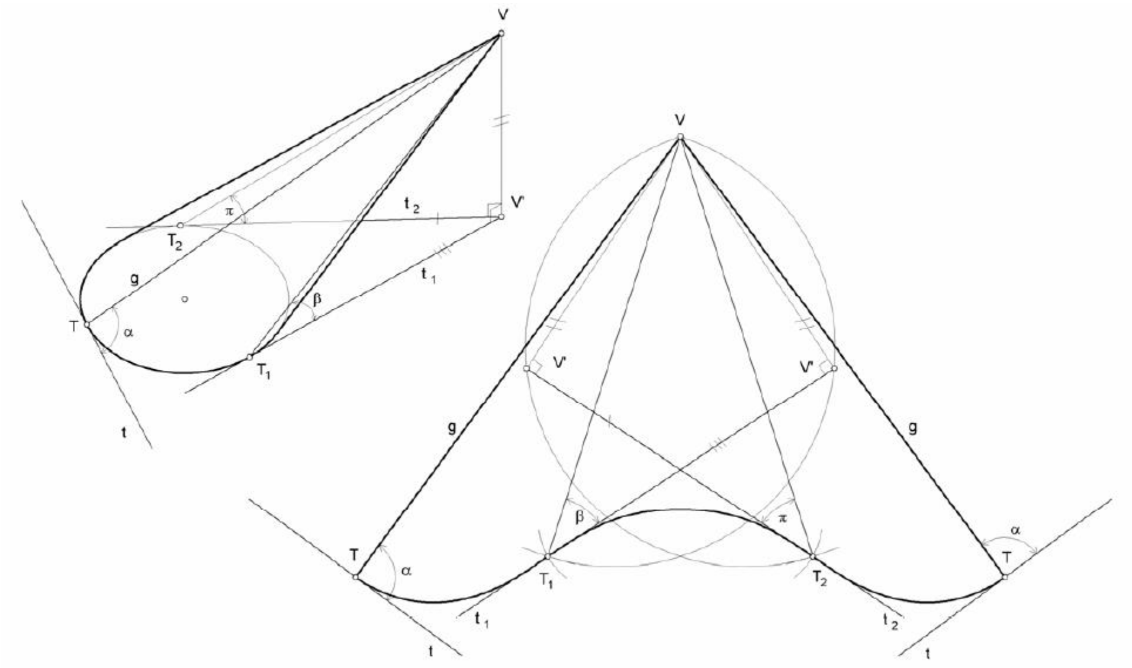


figure 50: Unrolling of the surface of the oblique cone.

3.2.3.7. Developing the surface of the tangential helicoid

Due to its regularity, the calculation of the unrolling of the surface the tangential helicoid is also relatively simple, as illustrated in the figure.

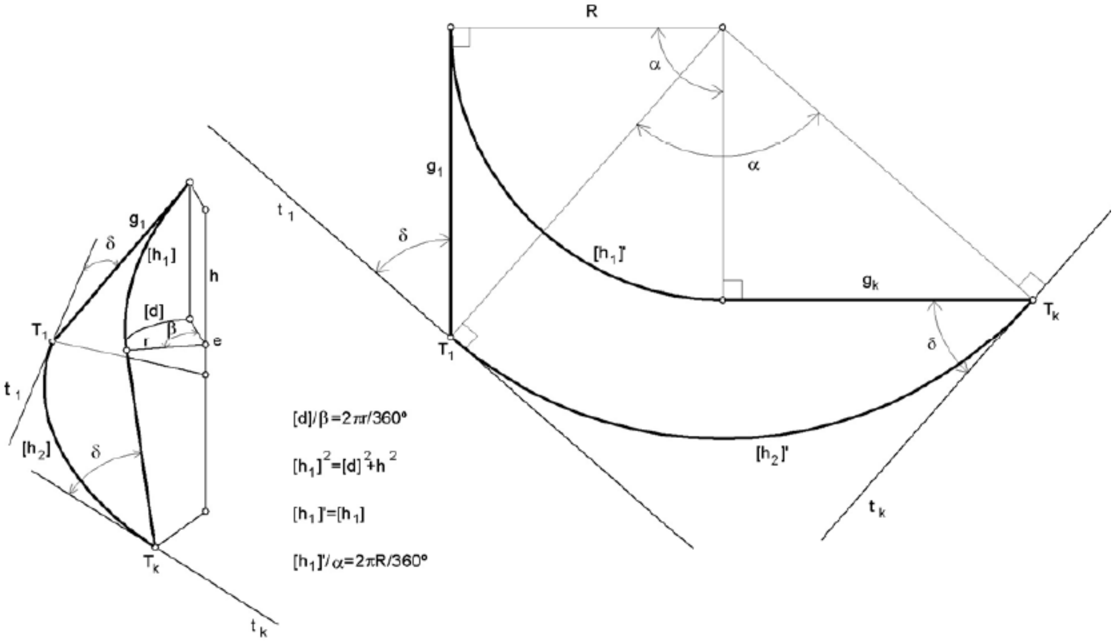


figure 51: Developing the tangential helicoid.

3.2.4. Warped surfaces

A ruled surface is not developable if two infinitely close generatrices do not intersect. This condition is generally met when the surface is defined by any three directrices. However, there are specific positions that the directrices may assume that do not allow the generation of any ruled surface or where it degenerates into a developable surface.

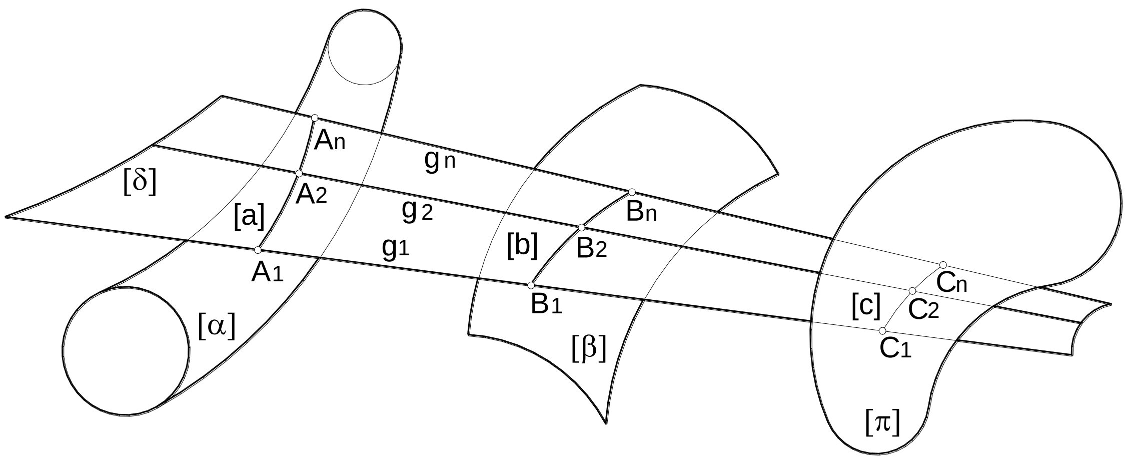


figure 52: Generation of a non-developable ruled surface.

The prerequisite for straight lines g_1, g_2, g_n to define a ruled surface is that they are tangent to the directrix surfaces $[\alpha]$, $[\beta]$ and $[\pi]$ simultaneously. That is, the surface $[\delta]$ must be simultaneously tangent to the surfaces $[\alpha]$, $[\beta]$ and $[\pi]$, along the lines $[a]$, $[b]$ and $[c]$, and respectively.

The set of lines g_1, g_2, g_n, is called the GENERATRIX SYSTEM.

If one of the directrix surfaces is replaced by a directrix line, then the generatrices must intersect it.

If the surface $[\delta]$ has only one generatrix system g_1, g_2, g_n, then it is said to be SIMPLY RULED.

If the surface $[\delta]$ has two generatrix systems g_1, g_2, g_n and j_1, j_2, j_n, then it is said to be DOUBLY RULED.

When a surface is doubly ruled, all generatrices in one system intersect all generatrices in the other system.

If a directrix is improper (located at infinity) this means that all generatrices g_1, g_2, g_n are parallel to one orientation. In this case the surface is said to be of DIRECTOR PLAN.

If a curved directrix is improper (located at infinity), this is to say that all the generatrices g_1, g_2, g_n are parallel to the generatrices d_1, d_2, d_n of a conical surface. In this case, the surface is said to be of DIRECTOR CONE.

However, it should be noted that even if the surface is defined by 3 directrices, it will necessarily have the property of being of director plane or of director cone, since all lines have improper points. In any case, in terms of classification as to the directrix, it is necessary to distinguish between those previously referred and the ORDINARY ones.

Therefore, the following table of classification of non-developable ruled surfaces defined by three directrices is given.

NON-DEVELOPABLE RULED SURFACES (WARPED SURFACES) DEFINED BY 3 DIRECTRICES (lines and/or surfaces) R (straight line) ; C (curve) ; S (surface) ; R∞ (improper straight line) ; C∞ (improper curve)	TYPE	DIRECTRICES	exemplos
	ORDINARY	R R R	Scalene ruled hyperboloid; One sheet revolution hyperboloid
		R R C	
		R C C	Skewed arc surfaces (cow's horn; "arriere-voussure")
		C C C	
		R R S	
		R C S	
		C C S	
		R S S	
		C S S	
		S S S	
	OF DIRECTOR PLANE	R∞ R R	Hyperbolic Paraboloid
		R∞ R C	Conoid surfaces; Helical surfaces
		R∞ C C	Cylindroid surfaces
		R∞ R S	Conoid surfaces with a core
		R∞ C S	Cylindroid surfaces with a core; Helical Surfaces with a core
		R∞ S S	Two-core cylindroid surfaces
	OF DIRECTOR CONE	C∞ R R	Tetrahedroid
		C∞ C R	Helical surfaces
		C∞ C C	One sheet revolution hyperboloid
		C∞ R S	
		C∞ C S	Helical Surfaces with a core
		C∞ S S	

Table 2: Non-developable ruled surfaces classification.

3.2.4.1. One sheet revolution hyperboloid

The one sheet revolution hyperboloid accumulates the double condition of being surface of revolution and a ruled surface. Indeed, the same surface can be generated by rotating a hyperbola or by rotating a straight line skewed to the axis.

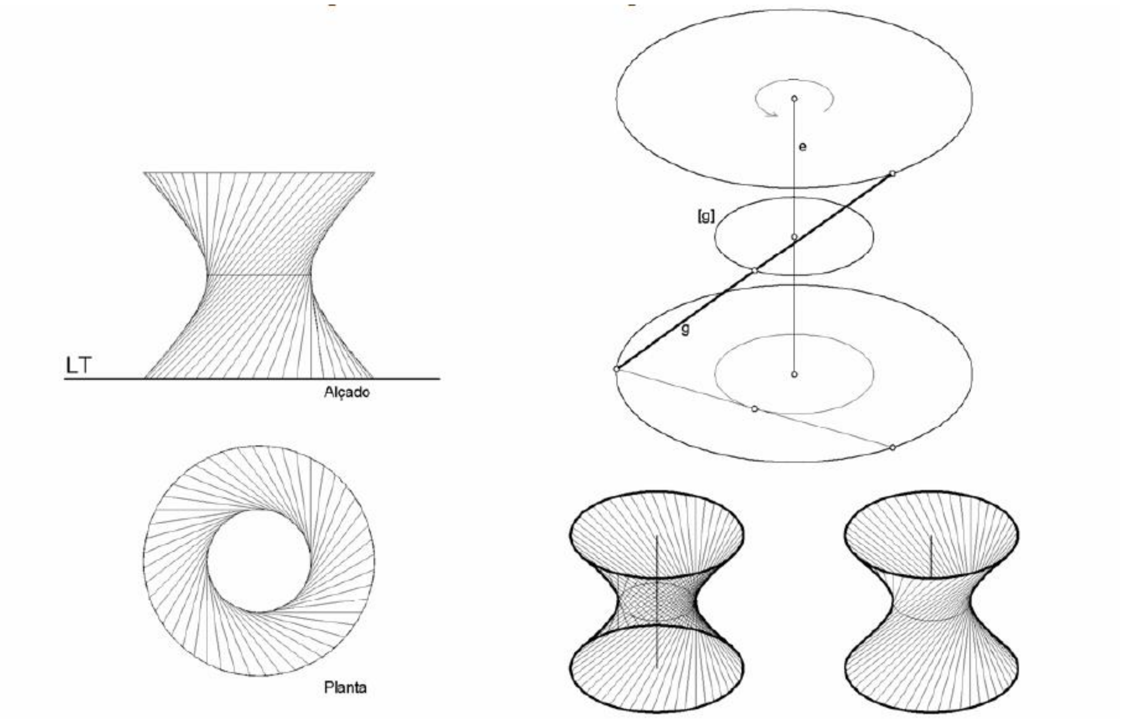


figure 53: One sheet revolution hyperbolid generated by the rotation of a straight line.

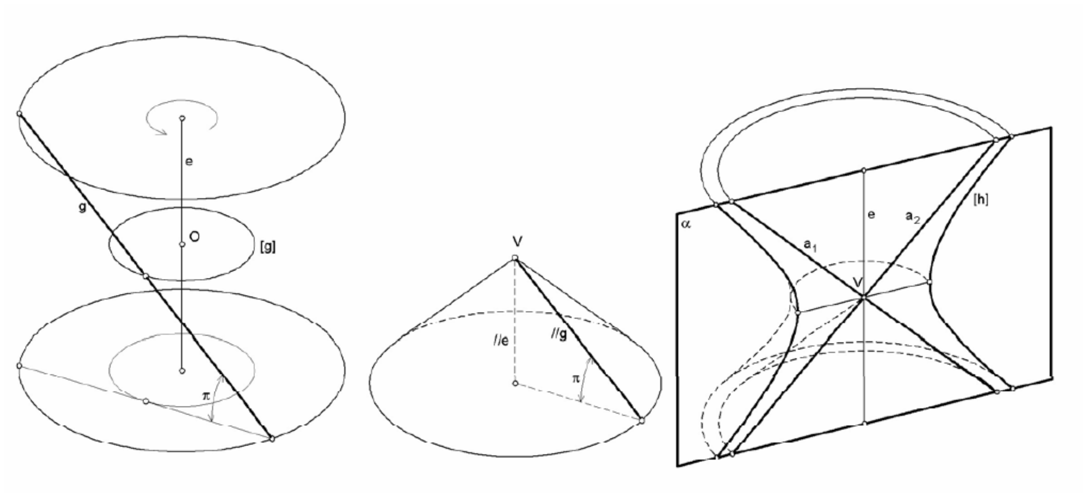


figure 54: Director cone and assimptotic cone of the one sheet revolution hyperboloid.

One of the interesting properties of the one sheet revolution hyperboloid is that its generatrices are parallel to those of a family of conical surfaces of revolution, the so-called director cones.

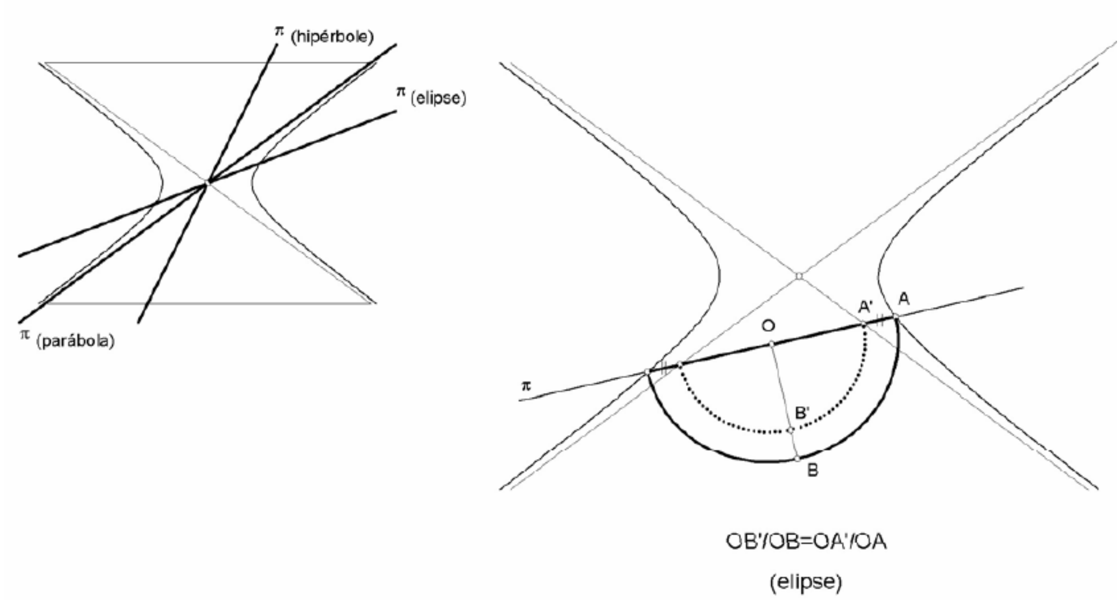


figure 55: Elliptical section on the one sheet revolution hyperboloid.

The intersections that can be produced on this surface are the same as those that can be produced on a conical surface of revolution, that is, they are conical lines. Planes with a certain orientation produce intersections of the same type as those produced in the director cones.

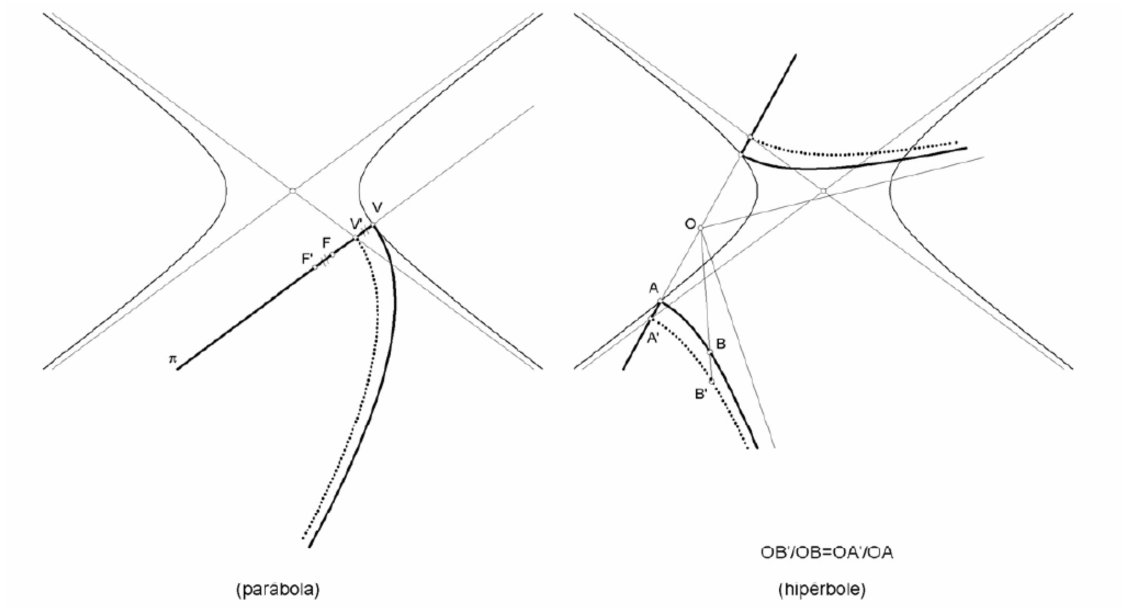


figure 56: Parabolic and hyperbolic sections of the one sheet hyperbolid of revolution.

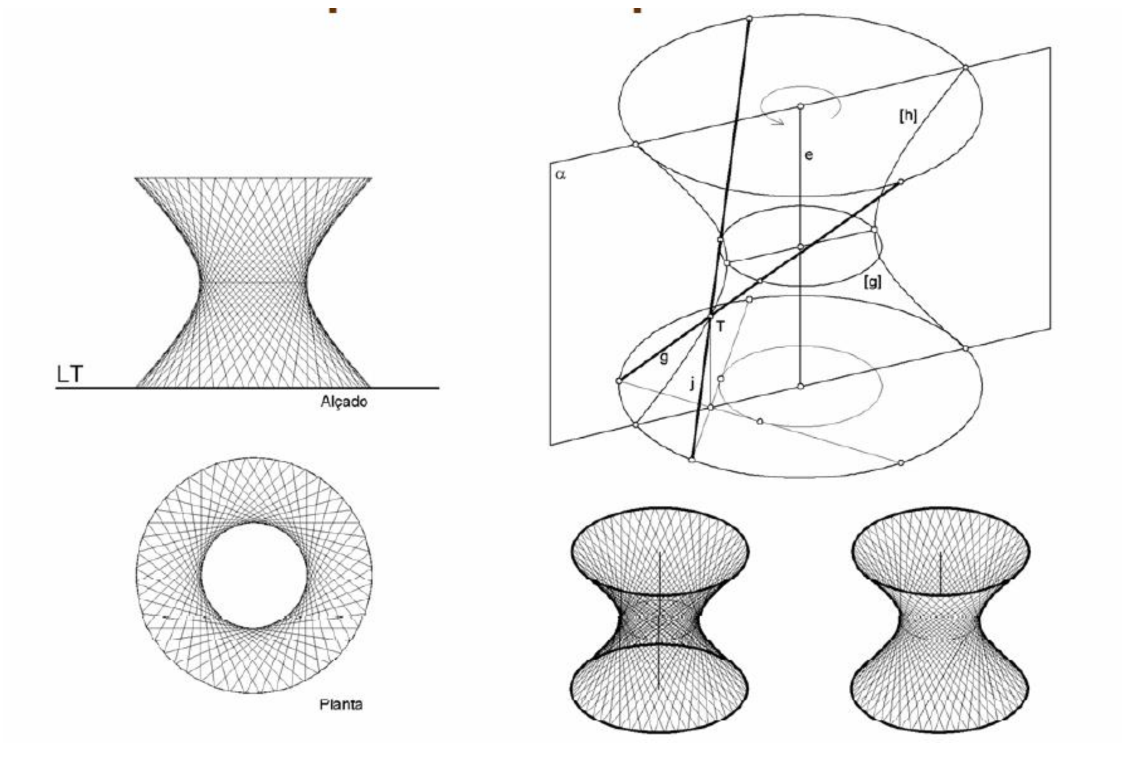


figure 57: The one sheet hyperbolid of revolution as a doubly ruled surface.

An interesting property of this surface is that it is doubly ruled, that is, the same surface can be generated in two different ways by rotating a line. Thus, this surface admits two generatrix systems, or in other words, the surface can be designed as a network of intersecting straight lines in space.

3.2.4.2. Scalene ruled hyperboloid

Scalene ruled hyperboloid can be obtained from the above by an affine transformation and enjoys properties very similar to that.

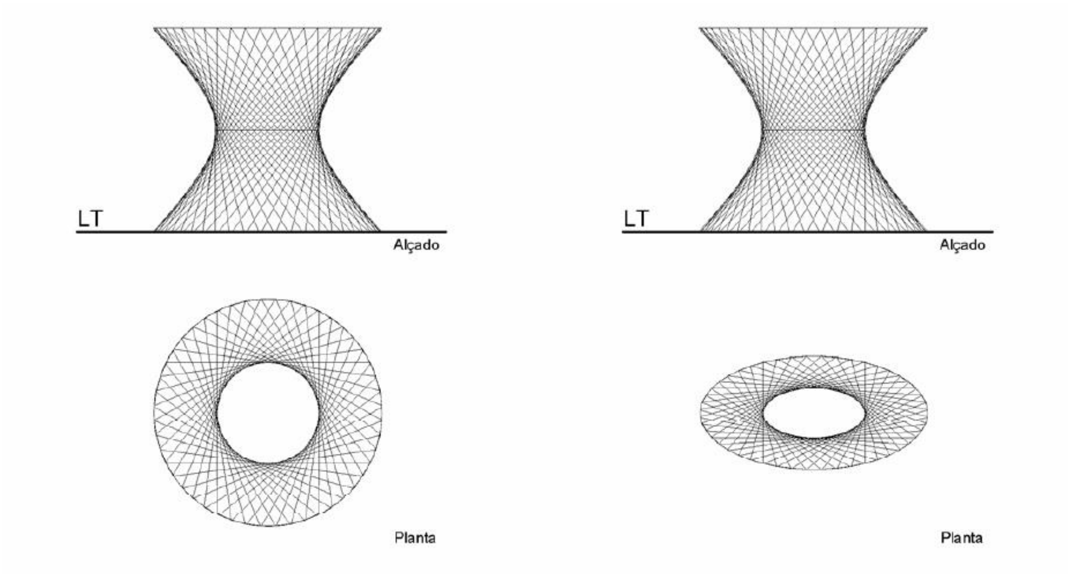


figure 58: One sheet revolution hyperboloid (left) versus Scalene ruled hyperboloid (right).

Another way to generate scalene ruled hyperboloid is as illustrated in the following figure. Of any three skew lines (directrices), the surface is generated by the movement of a fourth line (generatrix) that moves in space supported by those three. The generatrices of this surface can easily be obtained by intersecting a plane beam based on one directrix, with the other two directrices.

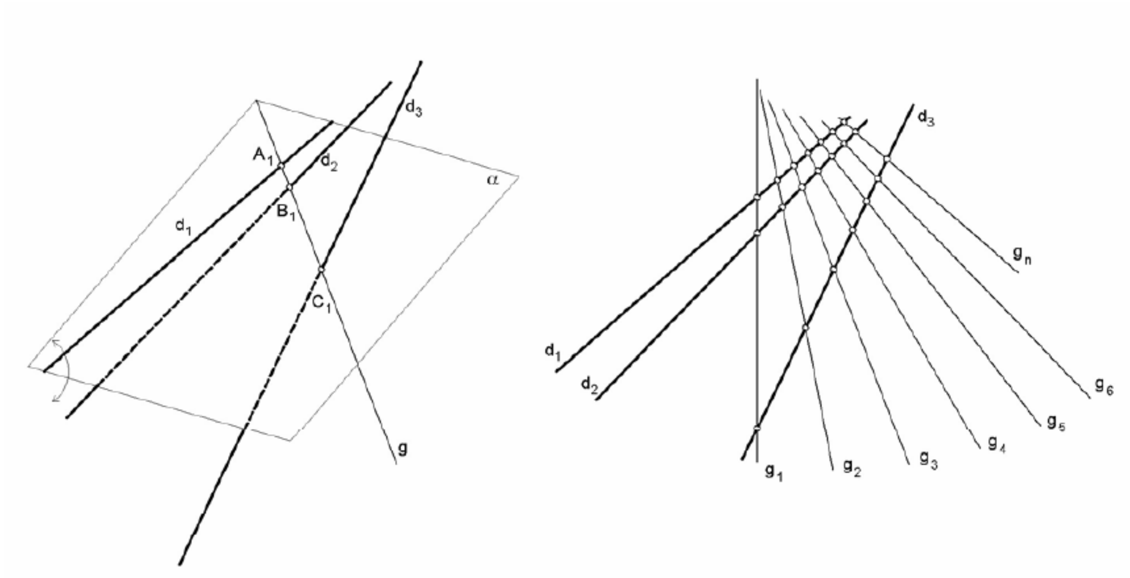


figure 59: Definition of the scalene ruled hyperboloid by three skewed directrices.

3.2.4.3. Hyperbolic paraboloid

Hyperbolic paraboloid can also be generated in several different ways. The following figure illustrates its generation by displacing a parabola (which preserves orientation) supported by another parabola.

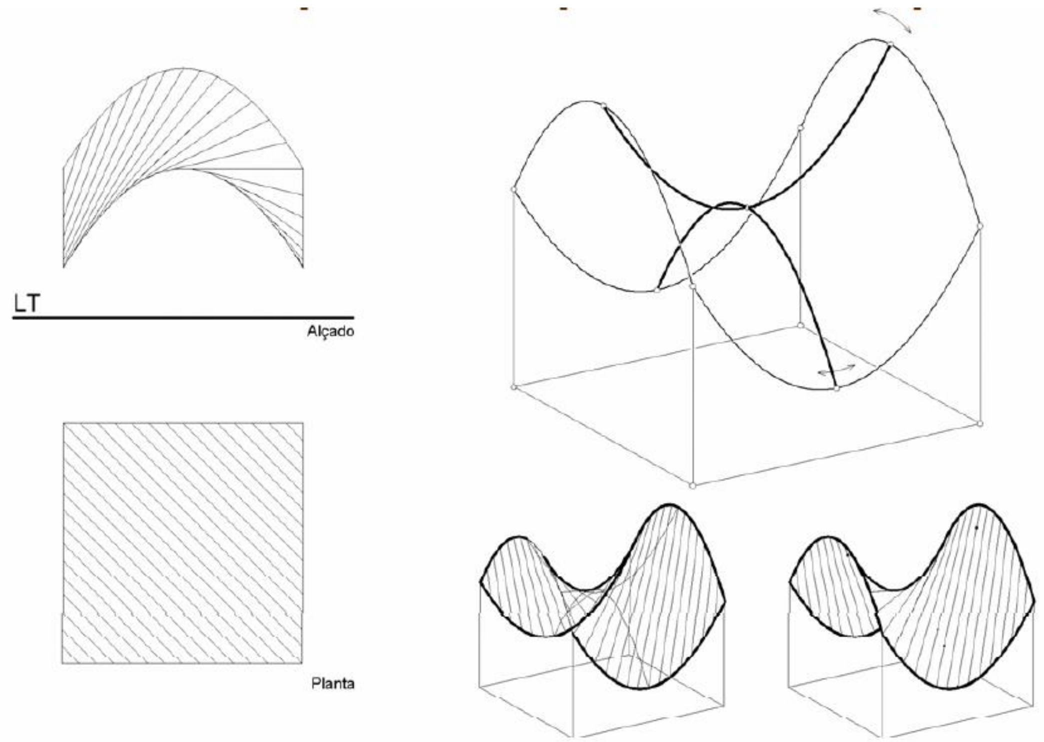


figure 60: Generating a hyperbolic paraboloid moving a parabola guided by another parabola.

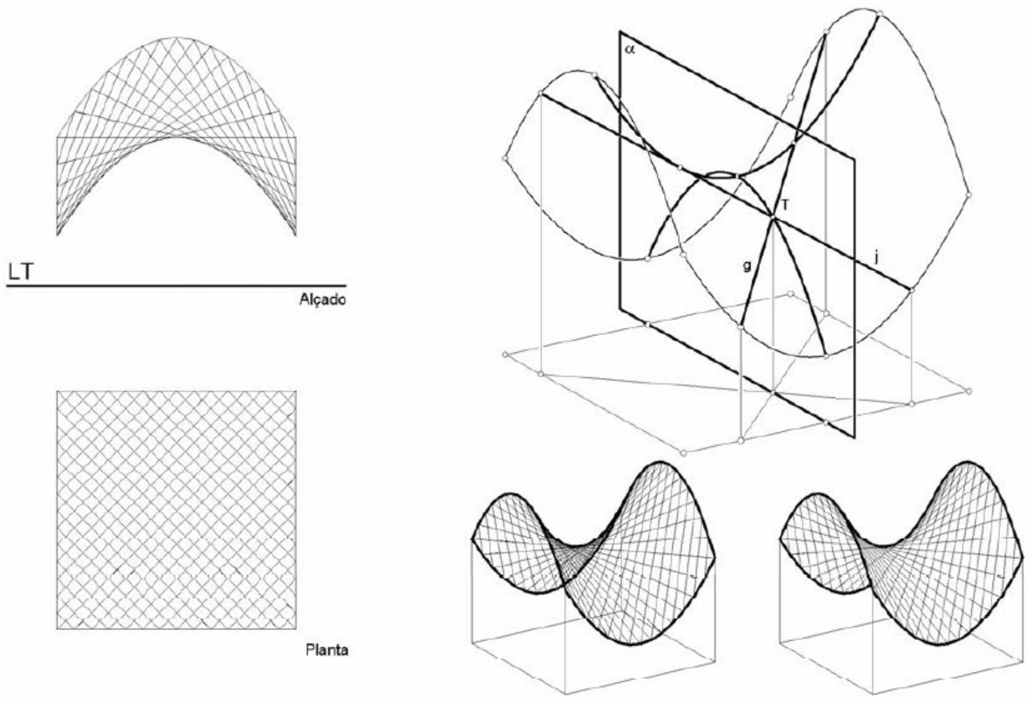


figure 61: The hyperbolic paraboloid as a doubly curved surface.

It is also a doubly ruled surface. That is, it can also be generated in two distinct ways by the movement of a line in space supported on three skewed lines. Although in this case one of the straight lines is improper. This means, in practice, that the generatrix moves by remaining supported by two skewed directrices and maintaining parallelism to a plan (plan orientation), called the director plan. Given the properties described, it appears that the hyperbolic paraboloid admits two orientations of director planes.

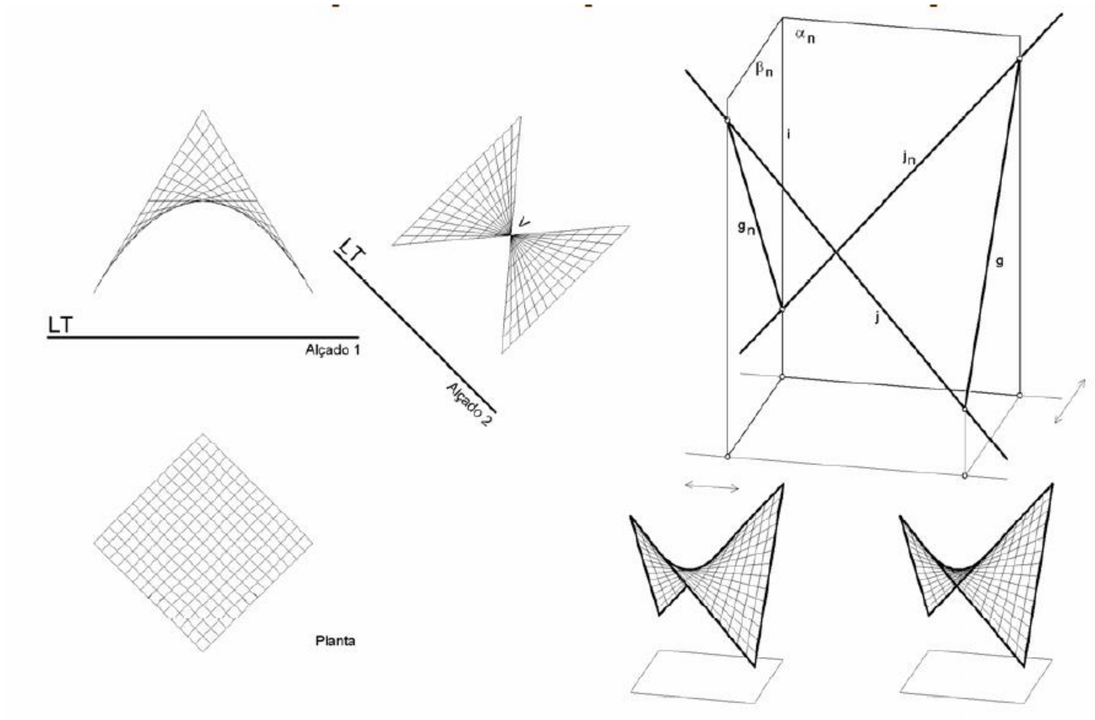


figure 62: The hyperbolic paraboloid; generation with straight lines; director planes; divergence point.

The simplest way to define a hyperbolic paraboloid is through a skewed quadrangle. The directions of two opposite sides define the orientation of the director plane of that family of generatrices, and the directions on the other two sides define the orientation of the director plane of the other family of generatrices.

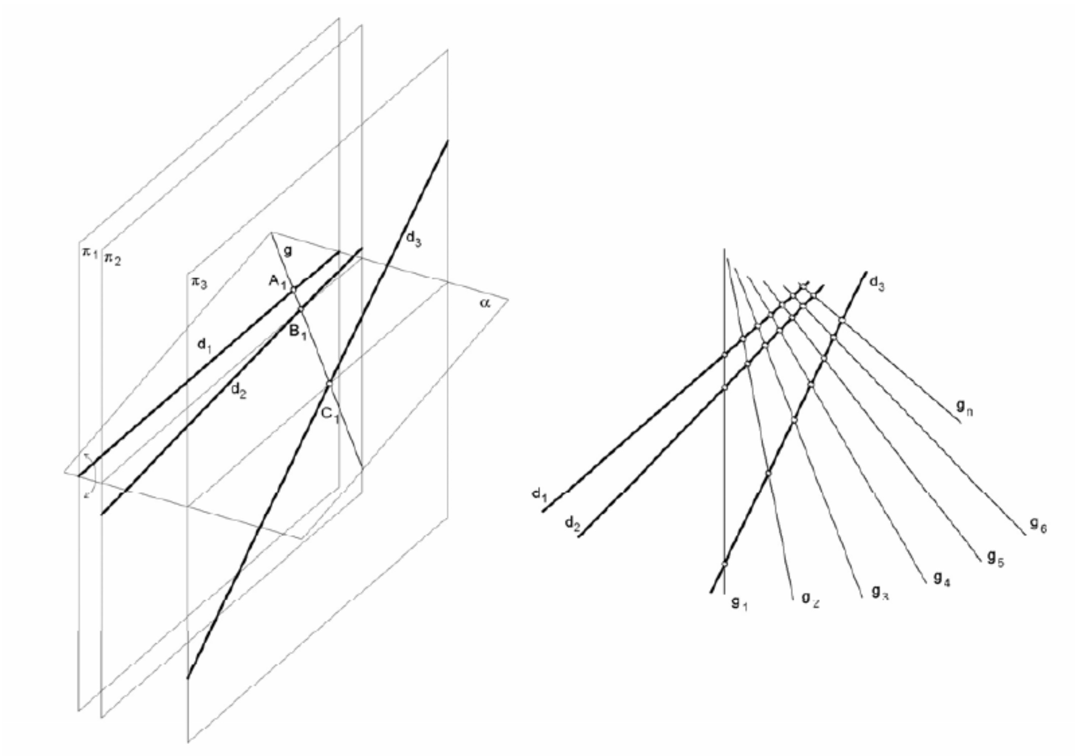


figure 63: Definiton of the scalene hyperbolic parabolid defined by three skewed straight lines.

To define the hyperbolic paraboloid given three generatrices of the same family, the condition must be that the directions of these lines are contained in a single orientation (the orientation of the director plane of that family of generatrices).

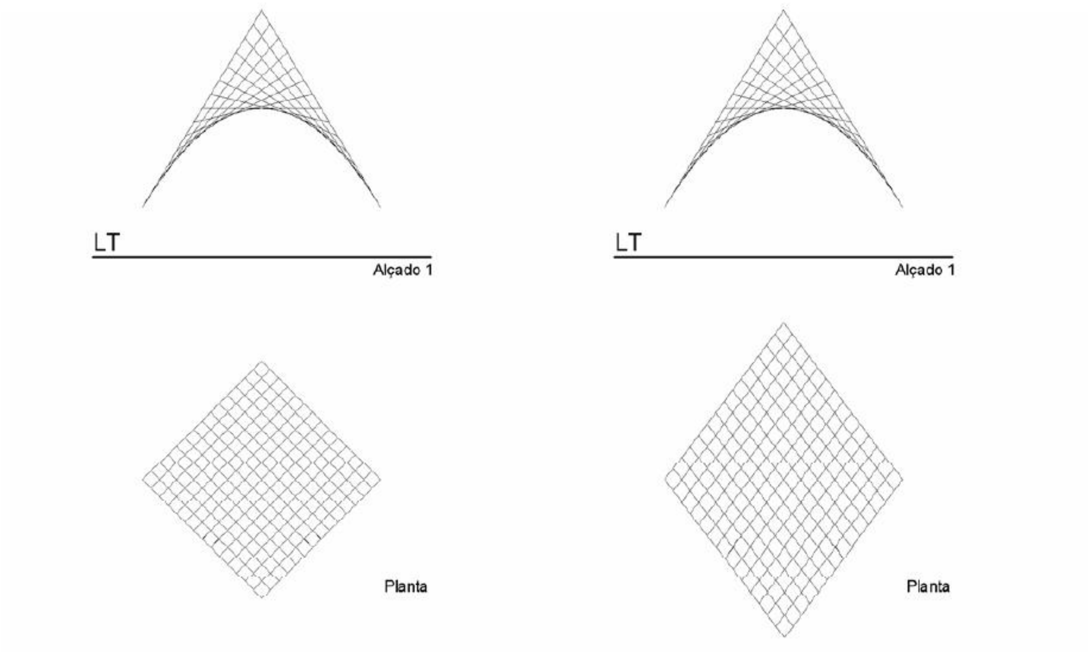


figure 64: Isosceles hyperbolic paraboloid (left) and scalene hyperbolic parabolid (right).

The hyperbolic paraboloid admits two types of conic sections. If a plane contains the direction common to both director planes, then the intersection is a parabola. Otherwise it is a hyperbola.

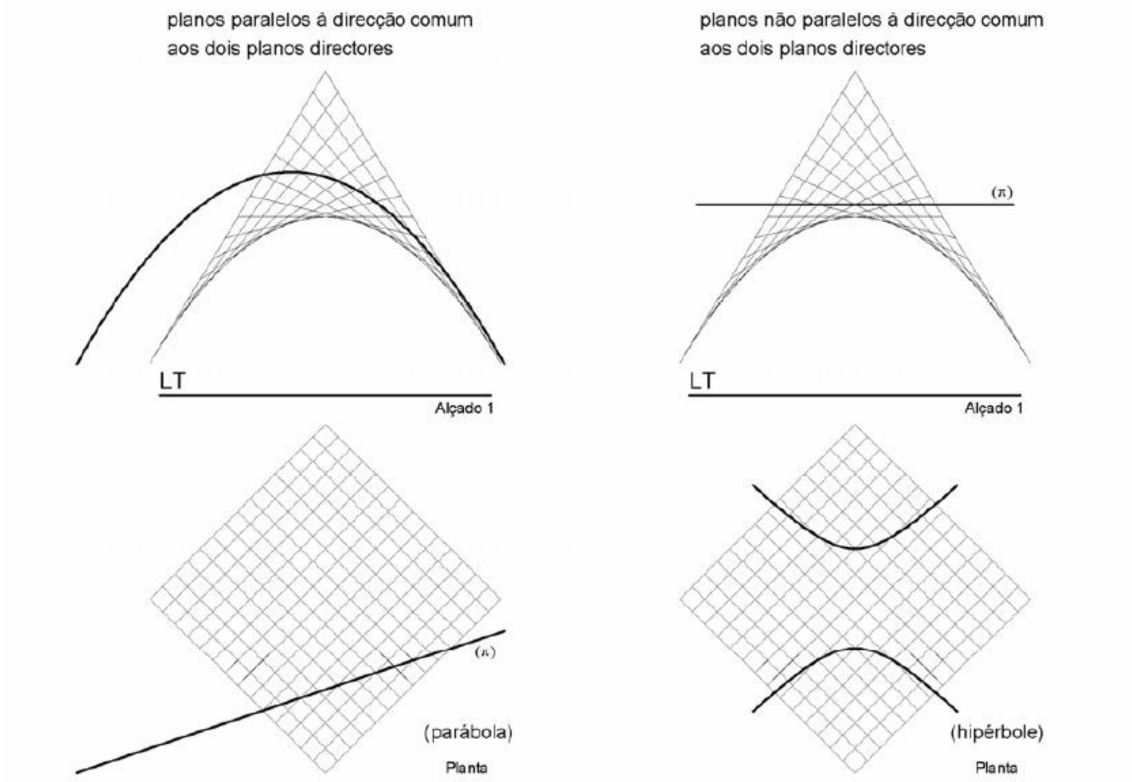


figure 65: Planar sections in the hyperbolic parabolid; parabola (left) and hyperbola (right).

Of course, we are not considering here the cases of tangent planes, where the sections degenerate into a pair of straight lines, which intersect at the point of tangency of the plane with the surface.

3.2.4.4. Ruled helicoids

Ruled helicoids are a family of non-developable (in general) ruled surfaces, which are guided by a cylindrical helix. They may be of director cone or director plan. They may have a central core or be axial.

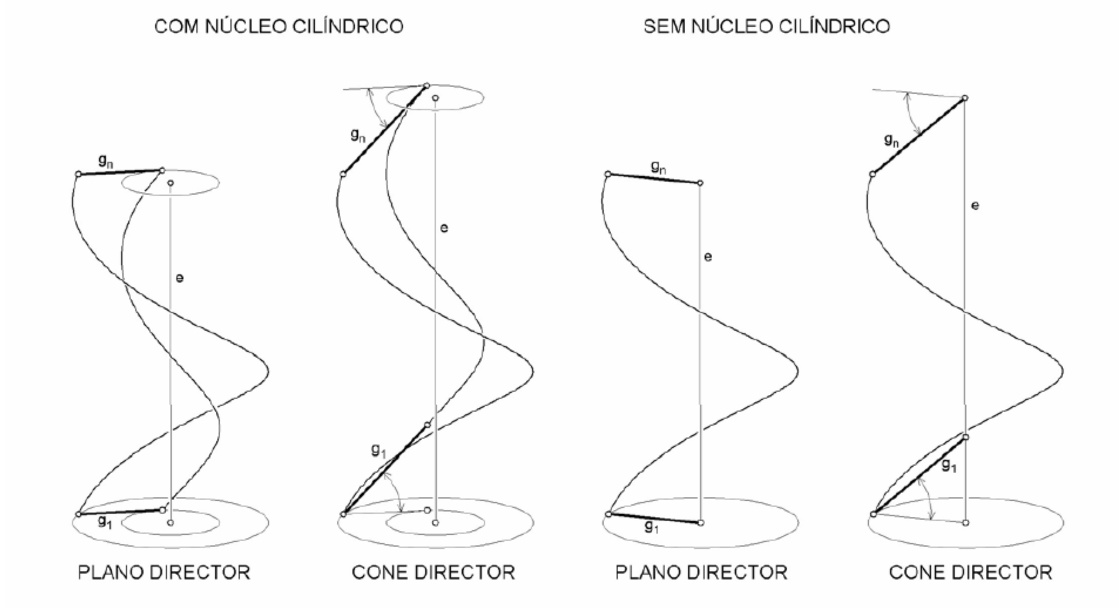


figure 66: Ruled helical surfaces generation.

When the helicoid is said to be axial, it means that the generatrices (considered extendible) are intersection the axis. In the other situation, the generatrices are tangent to the surface of a coaxial revolution cylinder with the helicoid. When they are of director plane, the generatrices retain a direction contained in an orientation, generally orthogonal to the surface axis. When they are of director cone, they maintain a constant angle with the axis, with the particularity that this angle is never 90° (this would be the case where there is a director plane).

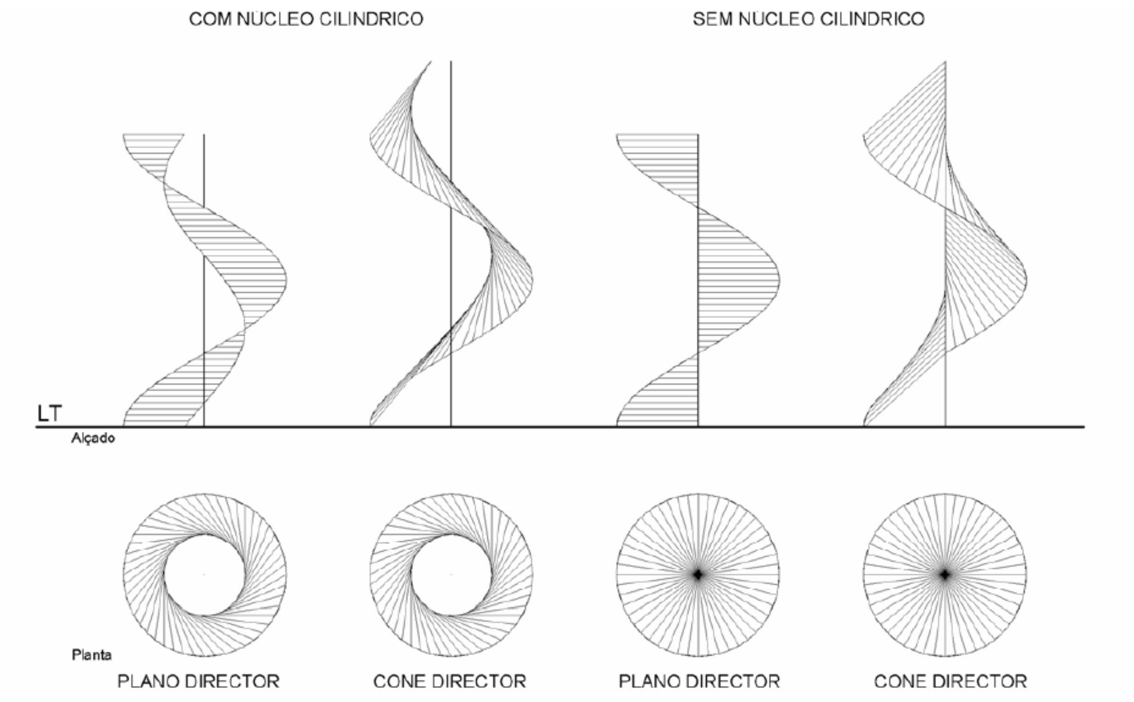


figure 67: Ruled helical surfaces (orthogonal views).

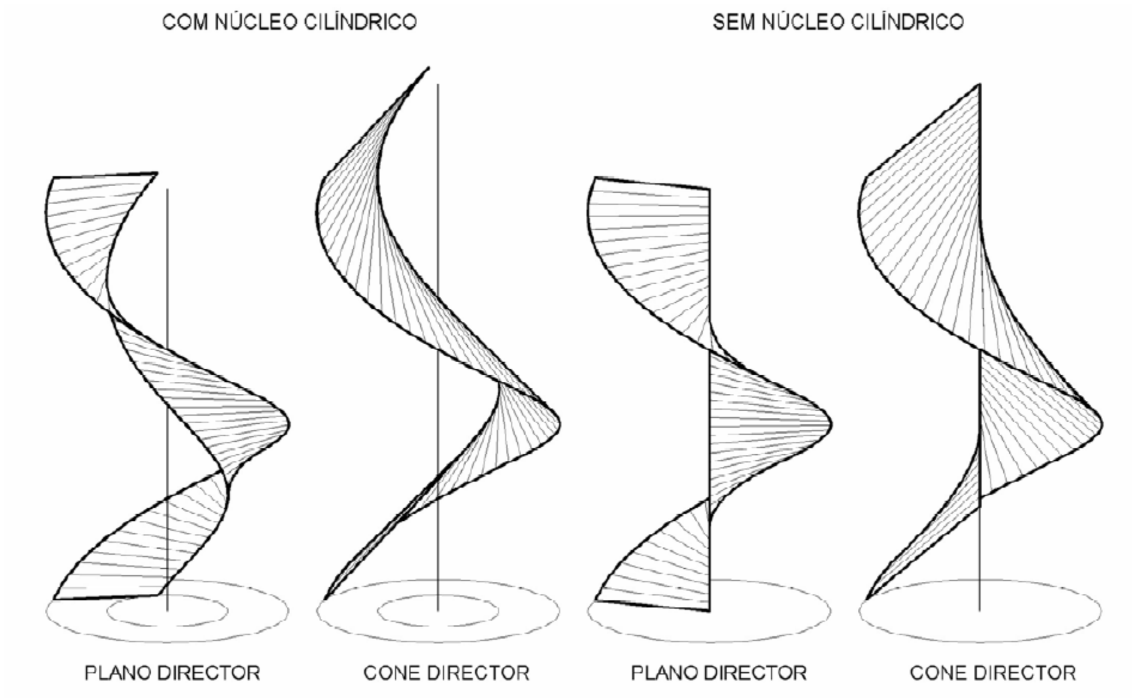


figure 68: Ruled helical surfaces (perspective views).

3.2.4.5. Conoid surfaces

Conoid surfaces can be generated by displacing a straight line (generatrix) that rests on another straight line and a curve, maintaining parallelism with respect to a plane orientation. That is, it is a simply ruled surface of director plane.

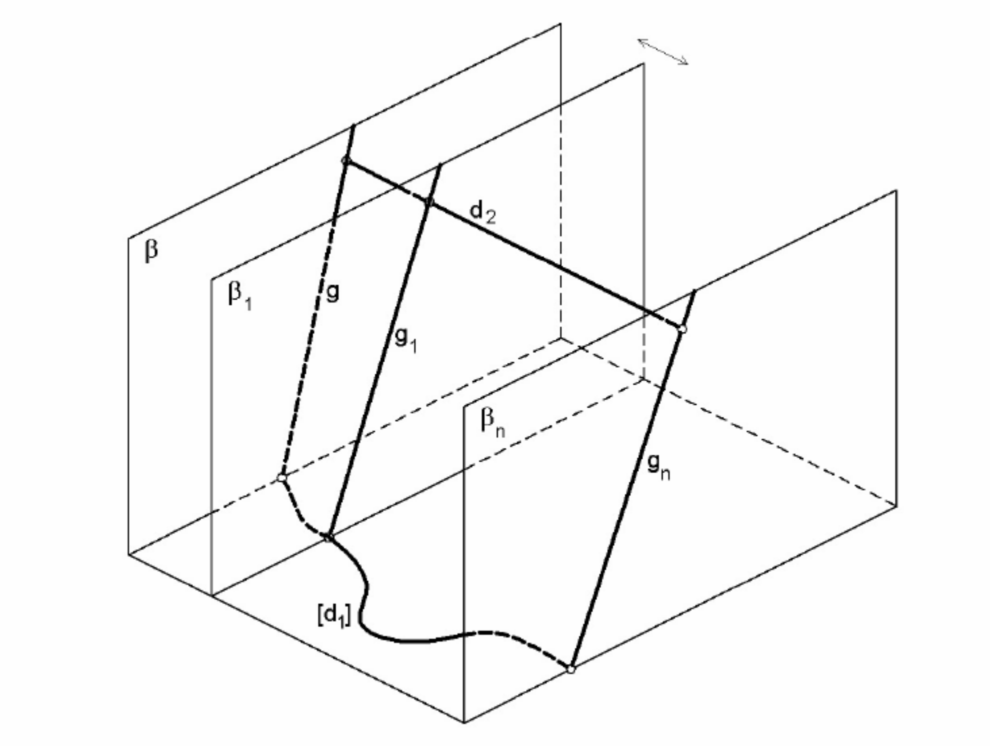


figure 69: Generation of a conoid surface.

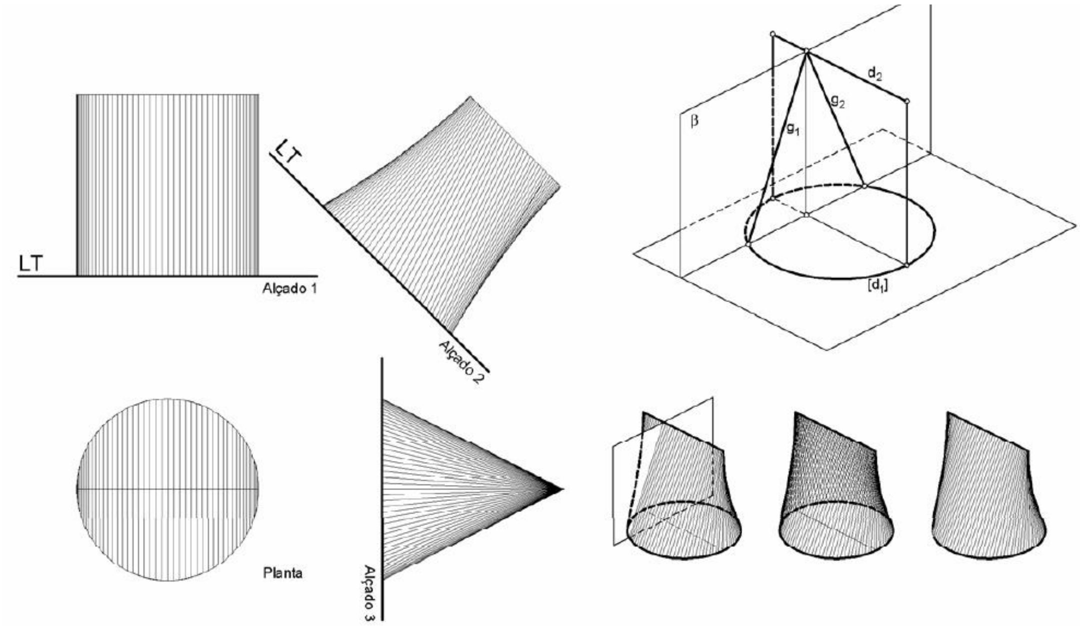


figure 70: Straight conoid with circumferencial directrix.

3.2.4.6. Cyilindroid surfaces

The cylindroid surface is like the previous one, but the generatrix rests on two curves. It is also a simply ruled surface.

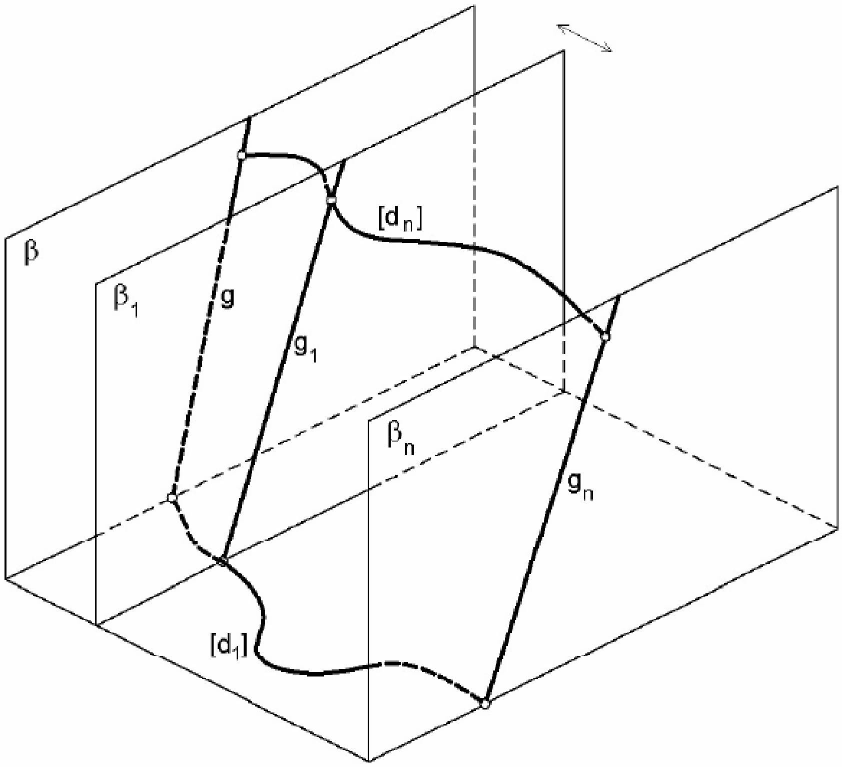


figure 71: Generation of the cylindroid.

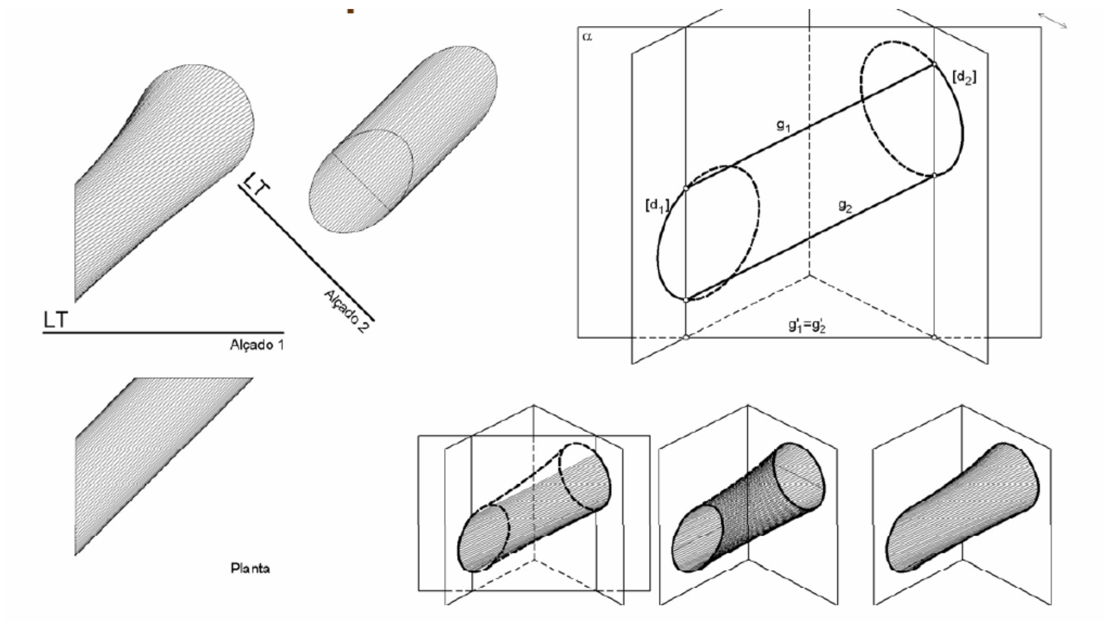


figure 72: Cylindroid with two circumferencial directrices.

3.2.4.7. Skewed arc surfaces

Skewed arc surfaces are generated by the lines defined by successive intersections of a plane beam relative to two curves or one curve and one line. The basis of the beam is one of the directrices of the surface.

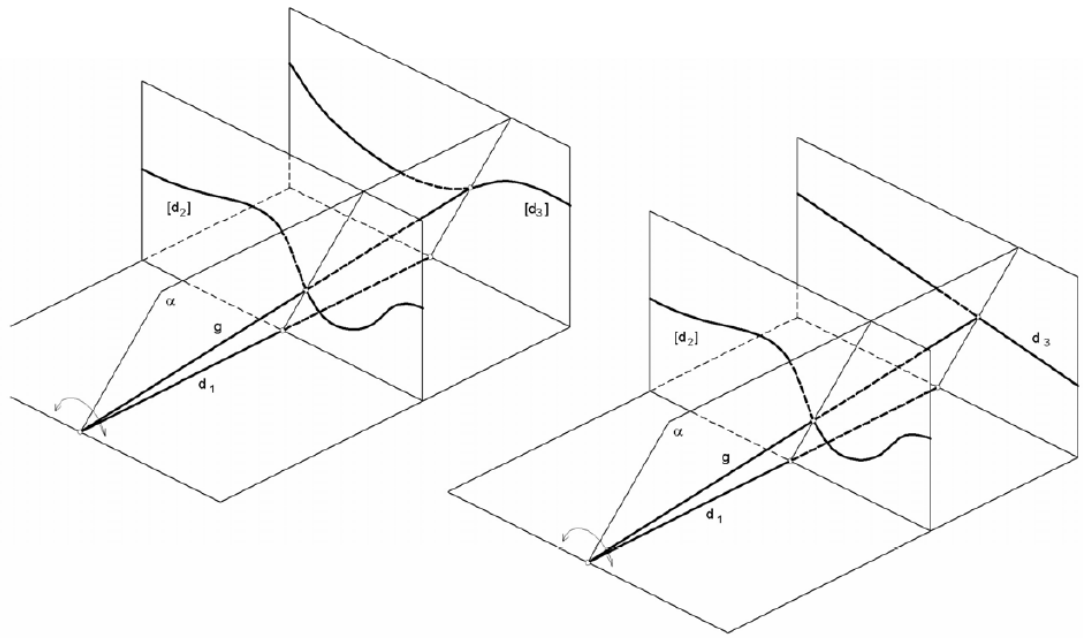


figure 73: Generation of the skewed arc surfaces.

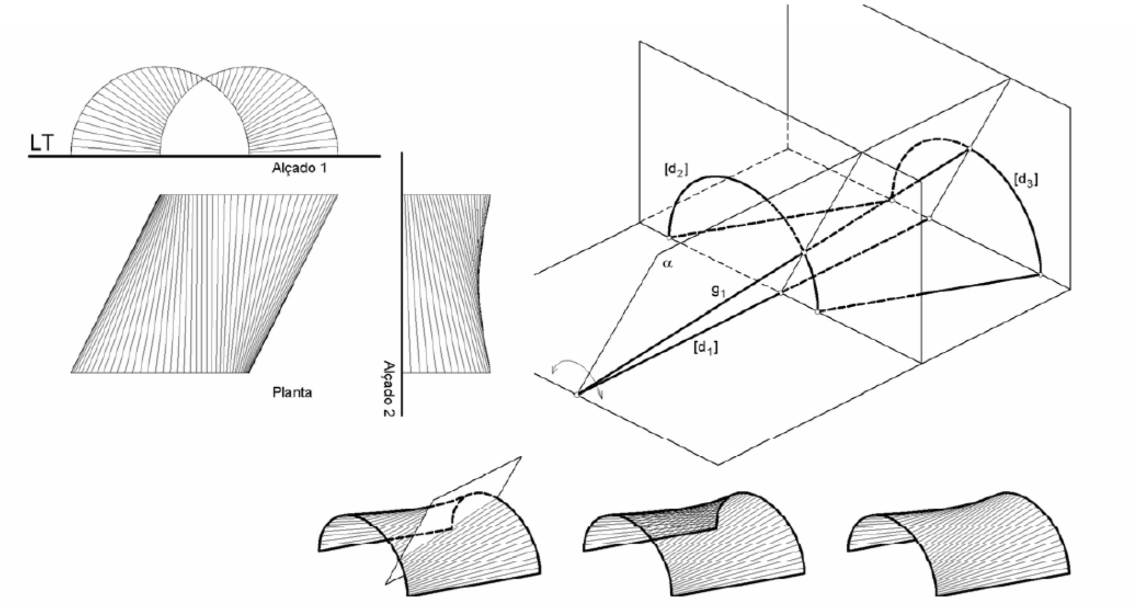


figure 74: The cow's horn, example of a skewed arc surface.

3.2.4.8. The tangent plane to a simply ruled surface

On a non-developable simply ruled surface $[\alpha]$, the plane π, tangent to $[\alpha]$ at a point T, contains the straight generatrix g passing through it. This plane intersects the surface along the straight line g and along a line $[a]$. The plane π contains the line t tangent to the line $[a]$ at the point T.

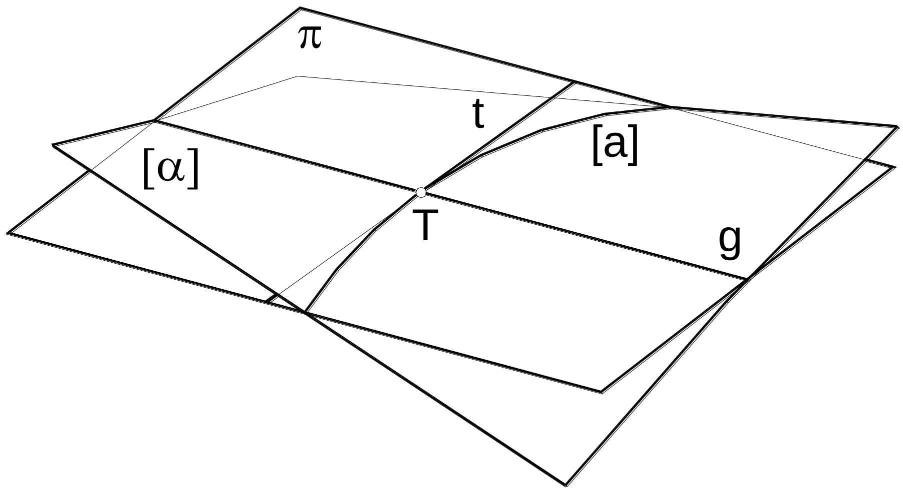


figure 75: The tangent plane to a simply ruled surface.

3.2.4.9. The tangent plane to a doubly ruled surface

On a doubly ruled surface, $[\alpha]$, the plane π, tangent to $[\alpha]$ at a point T, is defined by the two straight generatrices g and j, which are incident on it. This is the case of hyperbolic paraboloid, scalene hyperboloid and the one sheet revolution hyperboloid.

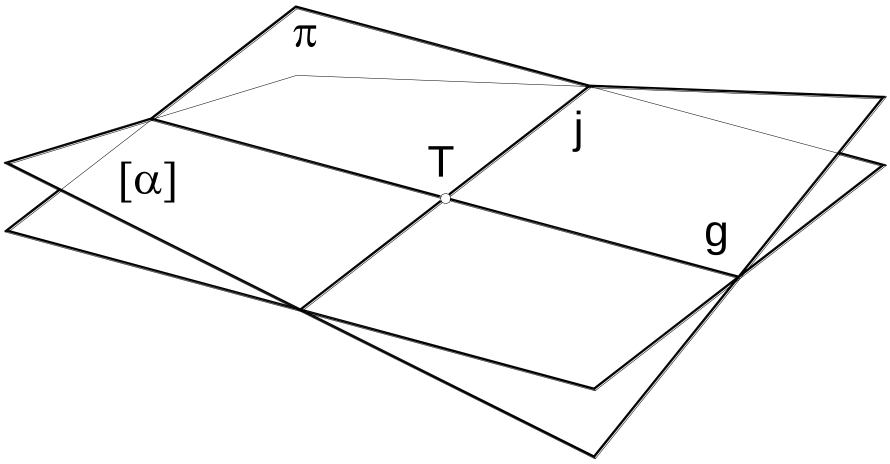


figure 76: The tangent plane to a doubly curved surface.

3.2.4.10. Bundle of tangent planes along a generatrix

Consider the ruled warped surface $[\delta]$ defined by the directrices $[a]$, $[b]$ and $[c]$.

Let g be a straight generatrix of the surface $[\delta]$, containing the points A, B and C, and belonging to the directrices $[a]$, $[b]$ and $[c]$, respectively.

The planes α_A, α_B and α_C, tangents to the surface $[\delta]$ at the points A, B and C, respectively, are defined by the generatrix g and the lines t_A, t_B and t_C, respectively tangent to $[a]$ at A, to $[b]$ at B and to $[c]$ at C.

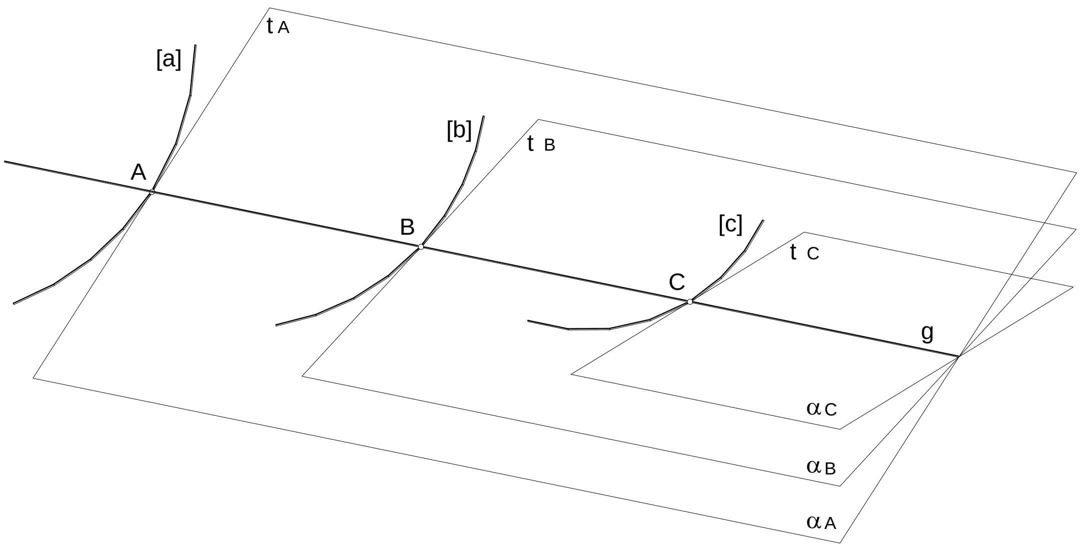


figure 77: Bundle of tangent planes along a generatrix.

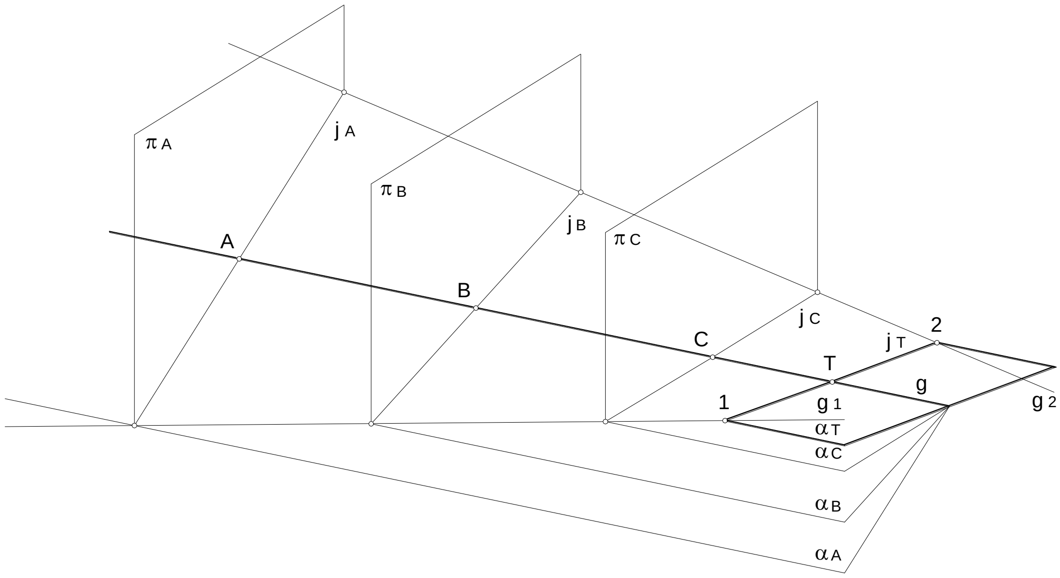


figure 78: Definition of a doubly ruled surface tangent to a simply ruled surface along a generatrix.

Following what was described in figure 77.

If you intersect the plane α_A with any plane π_A (passing through the point A), the plane α_B with any plane π_B (passing through the point B), and the plane α_C with any plane π_C (passing through the point C), you get the lines j_A, j_B and j_C, respectively, and tangent to the ruled warped surface $[\delta]$ at the points A, B and C, and respectively.

The three straight lines define a scalene hyperboloid tangent to the surface $[\delta]$ along the generatrix g.

As the planes π_A, π_B and π_C, and can assume a multitude of orientations, there is an infinitude of scalene hyperboloids tangent with the surface $[\delta]$ along the generatrix g.

If the three planes π_A, π_B e π_C are parallel to each other, the tangent surface is a hyperbolic paraboloid.

Once again, there is an infinitude of hyperbolic paraboloids tangent to the surface $[\delta]$ along the generatrix g.

Determining the plane α_T, tangent to the surface $[\delta]$ at any point T of any generatrix g, consists of determining the generatrix j_T (of the system opposite to g, and concurrent with g at the point T) of the scalene hyperboloid or the hyperbolic paraboloid, as the case may be.

3.2.4.11. Tangent planes to the one sheet hyperboloid of revolution

Since the surface of the hyperboloid of revolution is doubly ruled, the tangent plane is defined by the two generatrices of opposite systems that intersect at the point of tangency.

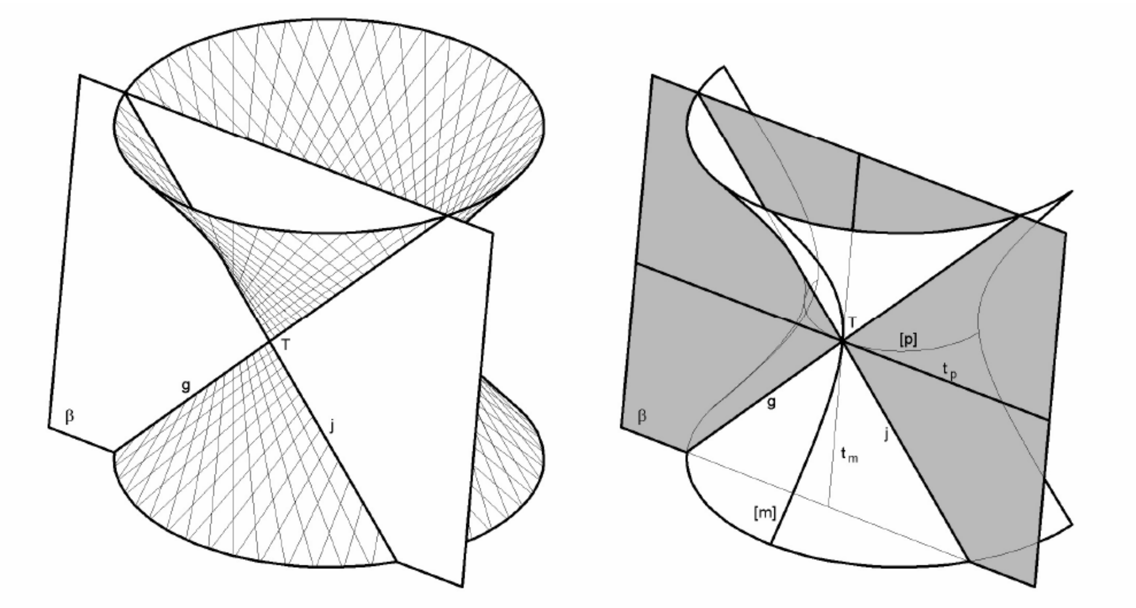


figure 79: Tangent plane driven by a surface point.

However, as with any surface, the plane can be defined by a pair of lines tangent to two intersecting sections at the point of tangency.

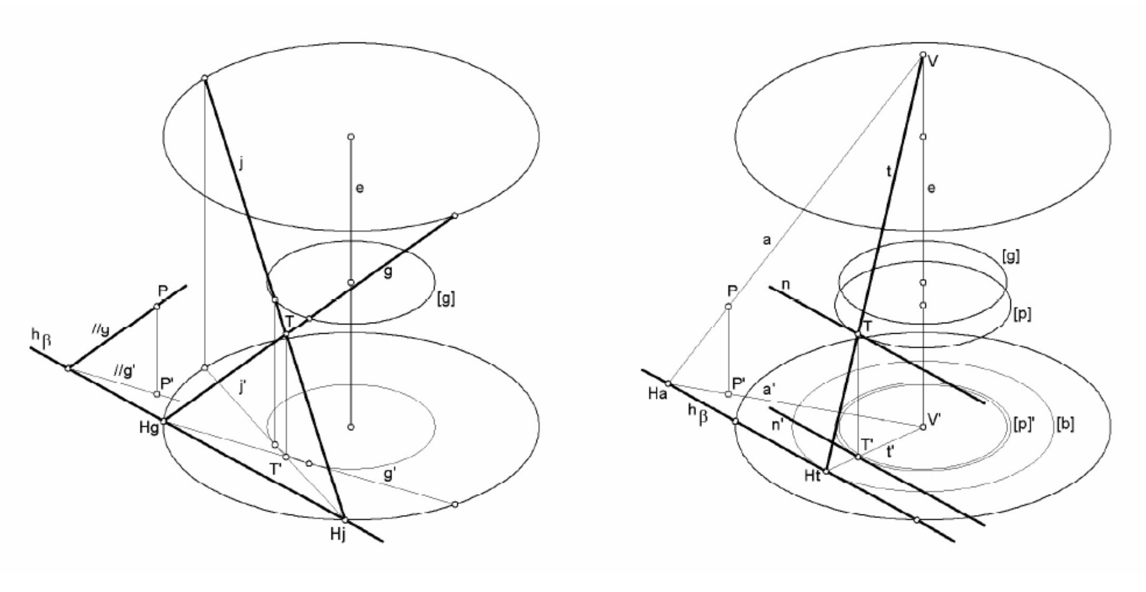


figure 80: Tangent plane driven by a point exterior to the surface.

Given a point exterior to the surface, it is possible to conduct a multitude of tangent planes to the surface. One way to restrict the number of solutions is to select a surface line (in principle straight) on which the tangent plane is determined. Thus the number of solutions is restricted to one or two in most cases.

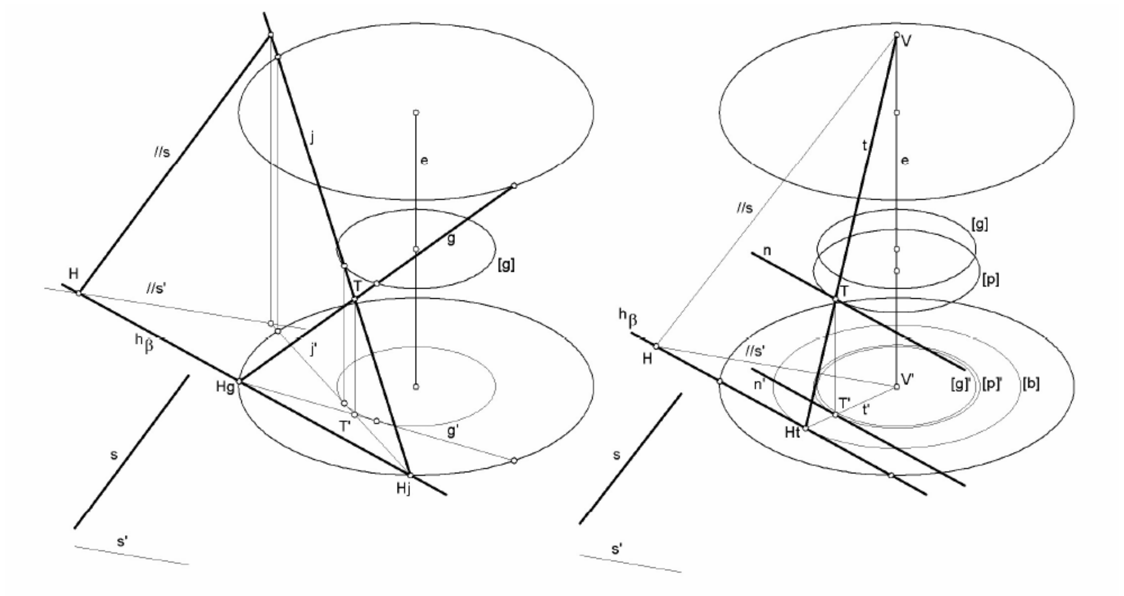


figure 81: Tangent plane parallel to a given line.

To drive the tangent plane parallel to a straight line implies to drive by any one of the generatrices, from one of the systems, an intersecting straight line with the given direction. The point of tangency is determined by identifying the generatrix of the other system contained in the plane.

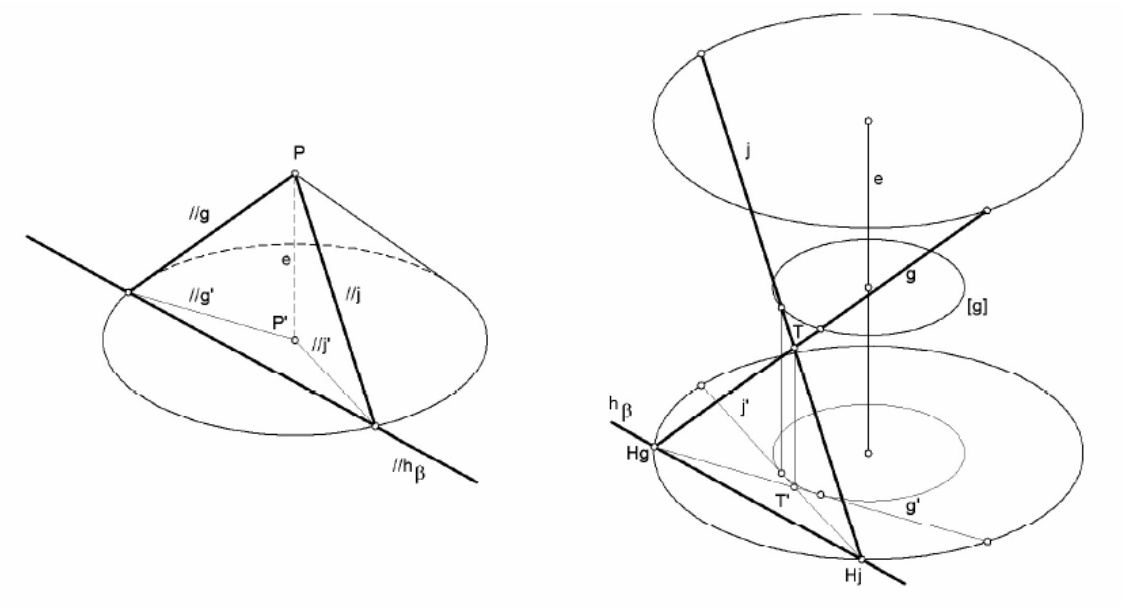


figure 82: Tangent plane parallel to a given plane.

To check whether it is possible to conduct a tangent plane parallel to a given plane, check whether there are directions in the director cone that are common to the plane. If so, the tangent plane is defined by the two generatrices with those directions, that is, the generatrices parallel to the given plane.

3.2.4.12. Tangent planes to the hyperbolic paraboloid

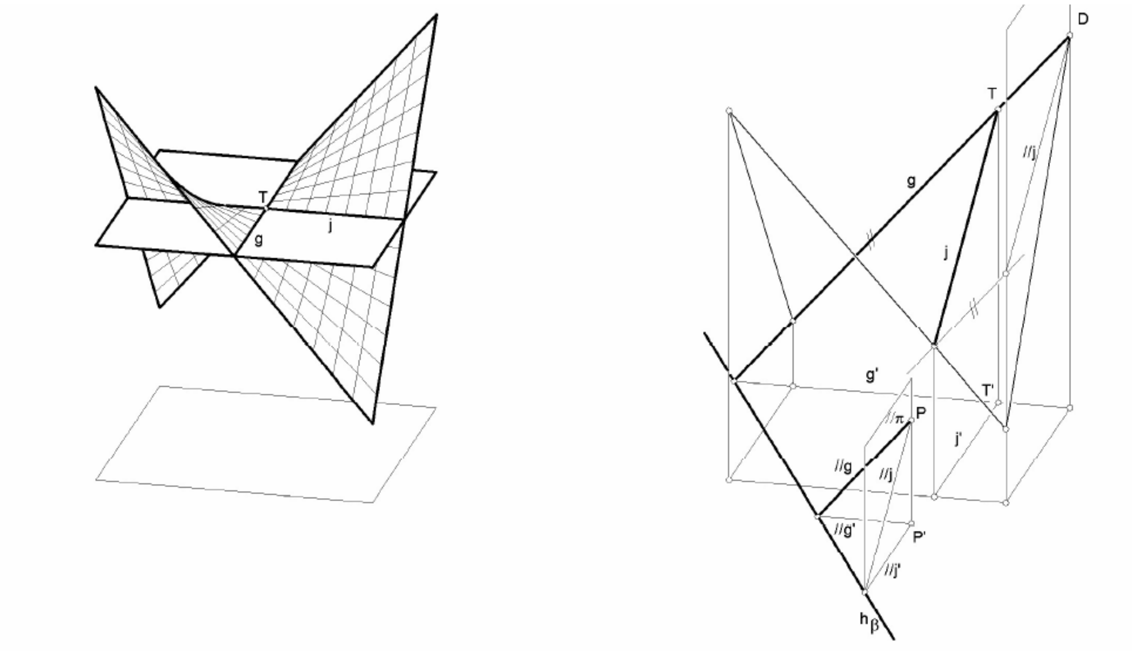


figure 83: Tangent plane driven by a surface point (left) and driven by an exterior point (right).

Also with the hyperbolic paraboloid, since it is a doubly ruled surface, the tangent plane at a point on the surface is defined by the two opposite-system generatrices that intersect at the point of tangency.

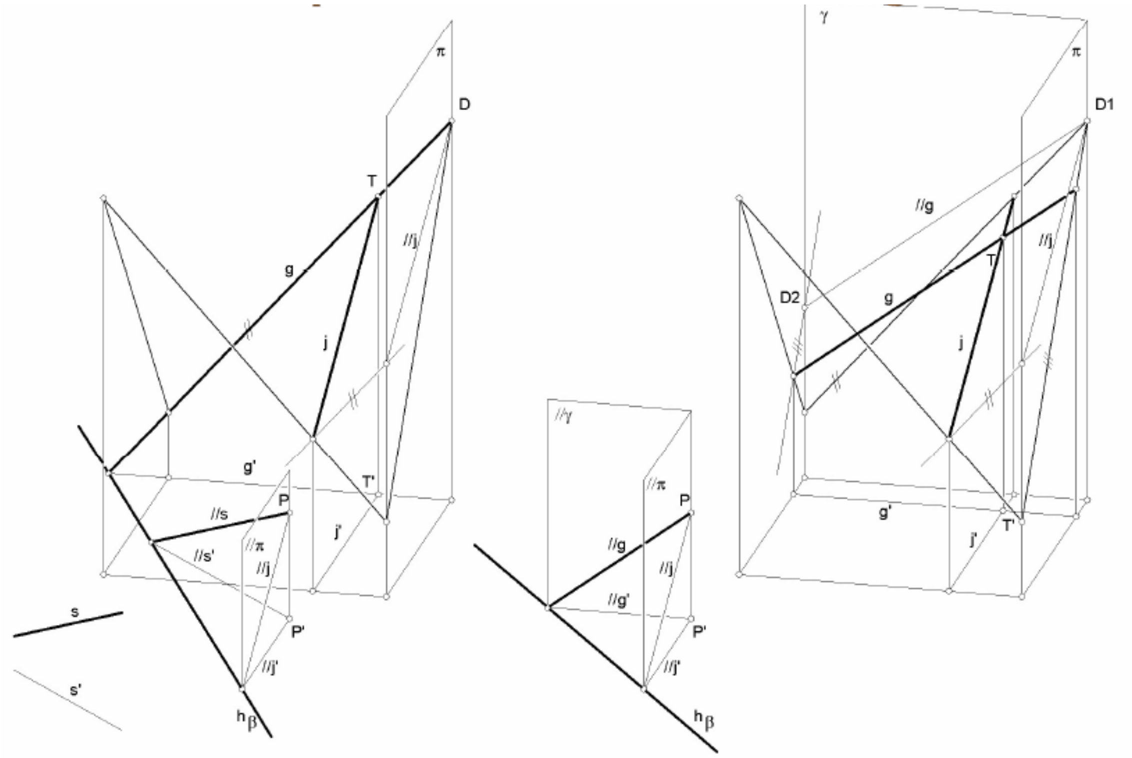


figure 84: Tangent plane parallel to a given line (left) and parallel to a given plane (right).

Determining the tangent plane parallel to a straight line or parallel to a given plane is all like what was said for the ruled hyperboloid. The difference now is the need to refer to the director plan instead of the director cone.

3.2.4.13. Tangent planes to the conoid

As stated in 3.2.4.8, the plane tangent to a simply ruled surface, driven by a point on the surface, contains the straight generatrix of the surface passing through the point. A further line may be obtained by passing at the point a tangent straight-line tangent to a surface that also passes through the tangency point. Or, it can be determined by trying to drive that line as a generatrix of a doubly-ruled surface tangent with the initial surface, since as already mentioned, two tangent surfaces share the same tangent planes along the line of tangency.

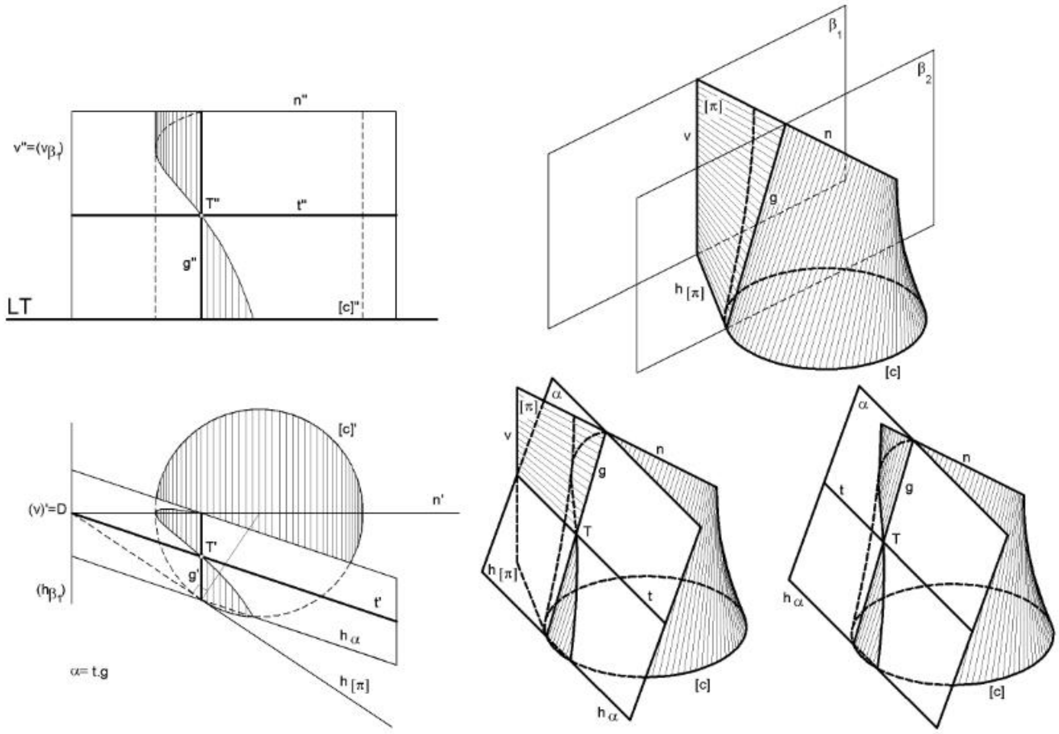


figure 85: Tangent plane driven by a surface point.

3.2.4.14. Tangent planes to the cow's horn surface

This surface appears to us here out of curiosity. The way in which it is treated is in all respects identical to the previous case.

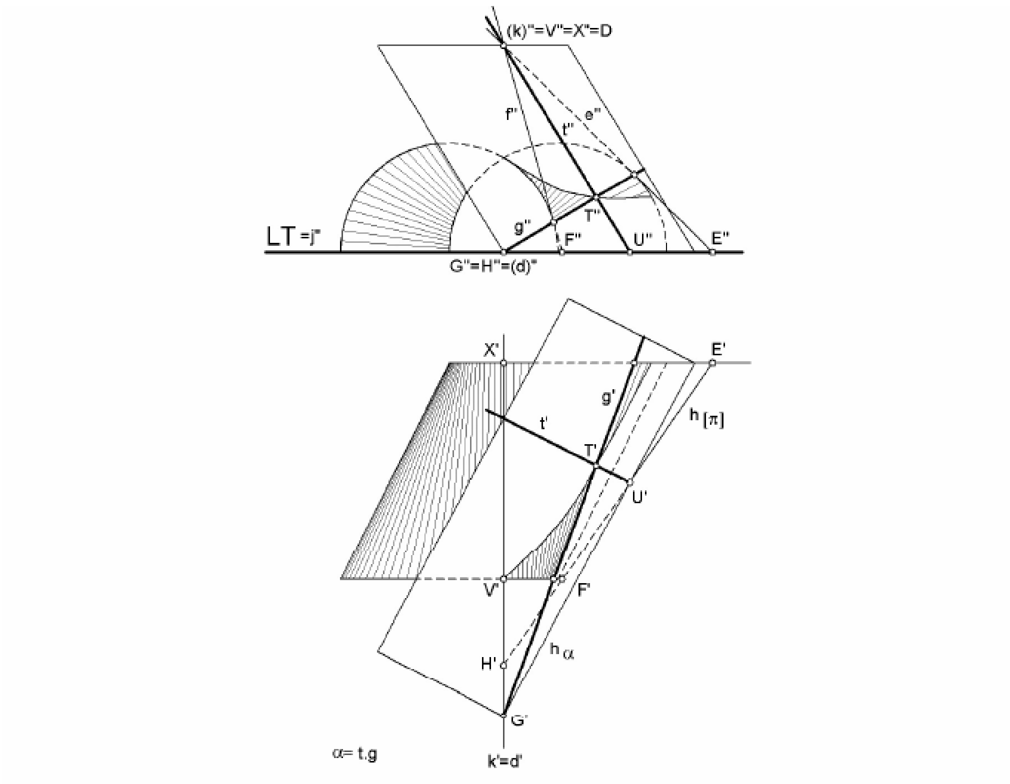


figure 86: Tangent plane driven by a surface point.

3.2.4.15. Tangency between surfaces as a composition tool

It is possible to move smoothly from one surface to another if they are tangent to each other, that is, if along the line common to both, the tangent planes are the same. In the two figures given below, there are tangencies between various surfaces and the hyperbolic paraboloid.

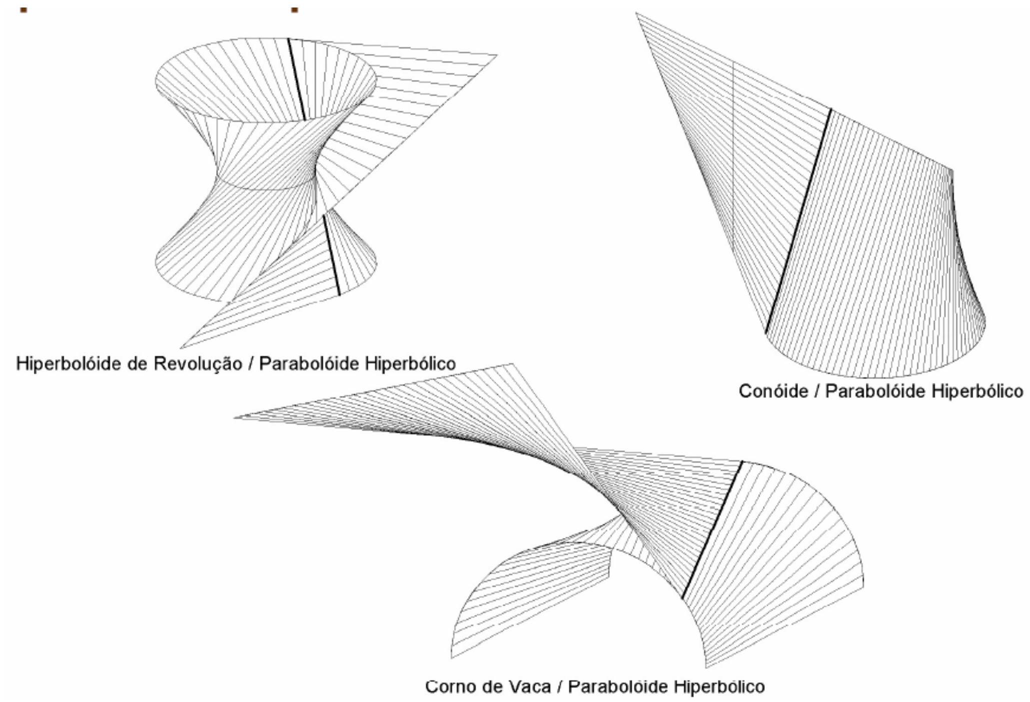


figure 87: Tangency between surfaces.

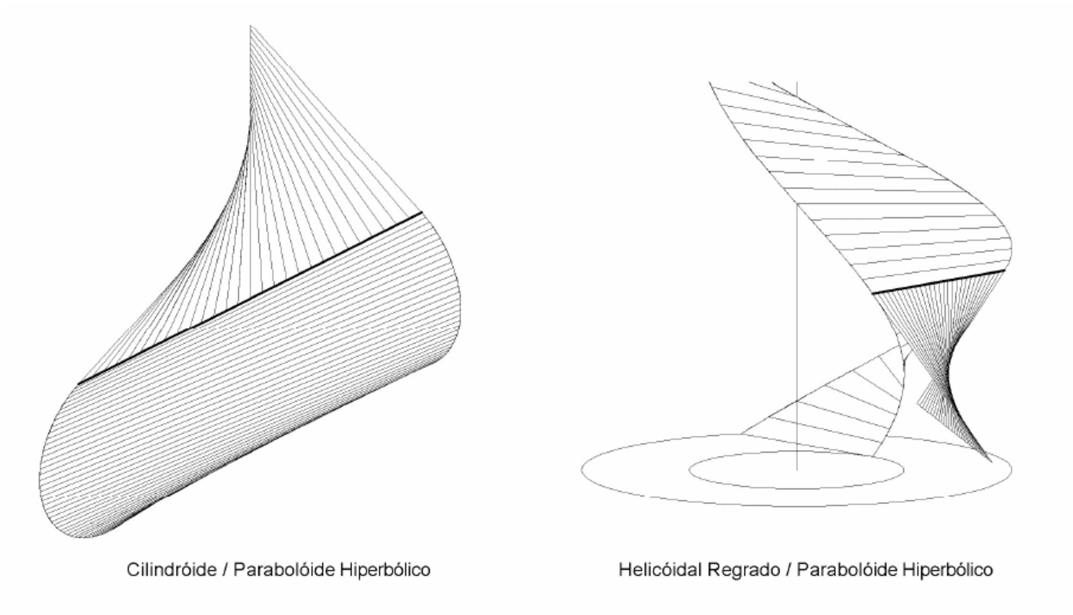


figure 88: Tangency between surfaces.

3.2.4.16. NURBS lines and surfaces in space

Everything that has been said about Bezier, B-splines, and NURBS lines in the plane can easily be expanded into three-dimensional space. For example, the control polygon line segments of a Bezier curve of degree three do not all have to be in the same plane. And the same applies to B-Splines and NURBS. This is a way of generalizing the representation of curves in space.

These curves, whether flat or not, spatially placed can serve as directrices for the generation of NURBS surfaces.

In the previous point, a whole series of surfaces has been discriminated down. Interestingly, you will probably not find many of them as geometric primitives in many 3D modelling tools. One way of representing those surfaces is by approximation through NURBS surfaces. Note, however, that NURBS surfaces, with the appropriate weights assigned to the control points, make it possible to accurately (geometrically) represent spheres, paraboloids, etc. The following two figures show the effect of changing the weight of the surface control network vertices.

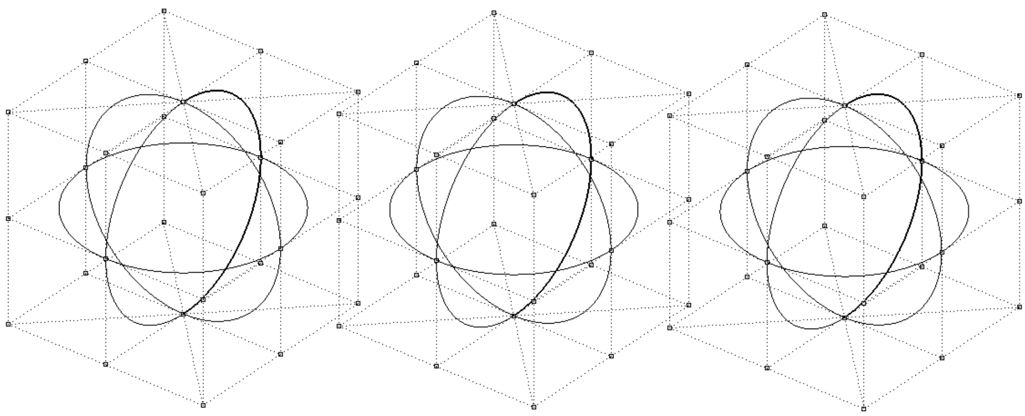


figure 89: Three control networks.

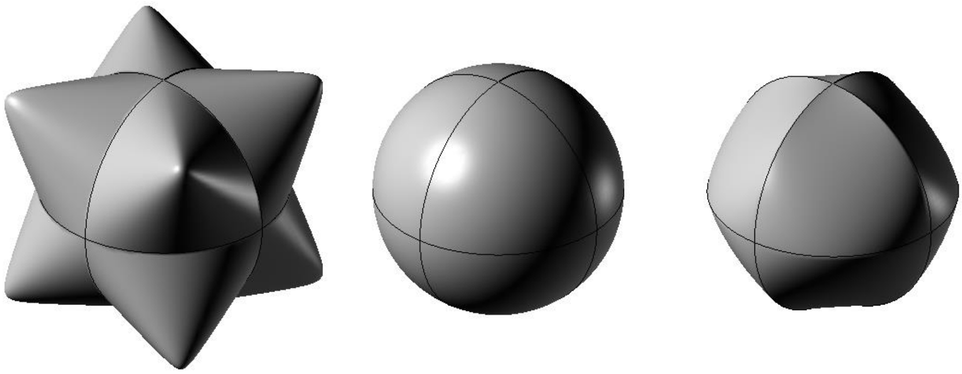


figure 90: The effect of changing the weight at the vertices of the control network.

In relation to the central figure (where a spherical surface is represented) the weights relative to the control cube vertices were increased (left surface) and the weights of the same vertices decreased (right surface). This is possible because the surface is represented as NURBS.

Generally, a NURBS surface is defined by a grid of control points, and their associated weights, and by the degree (and order) of their generatrices in the two parametric directions of the surface. These two parametric directions, usually noted by the letters U and V, can be visually represented by a grid of ISOPARAMETRIC curves on the surface.

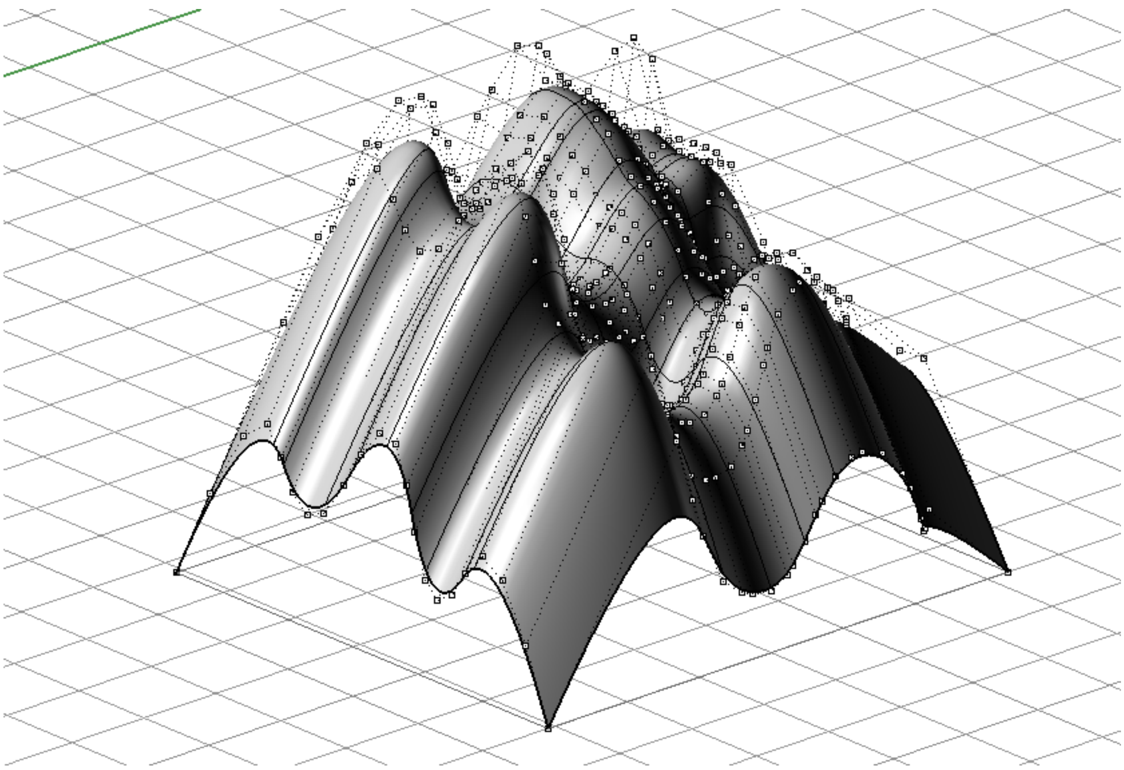


figure 91: NURBS surface and the respective control network vertices.

An isoparametric curve is a curve that, in one of the parametric directions, corresponds to a constant value of a parameter in parametric space (which is distinct from geometric space).

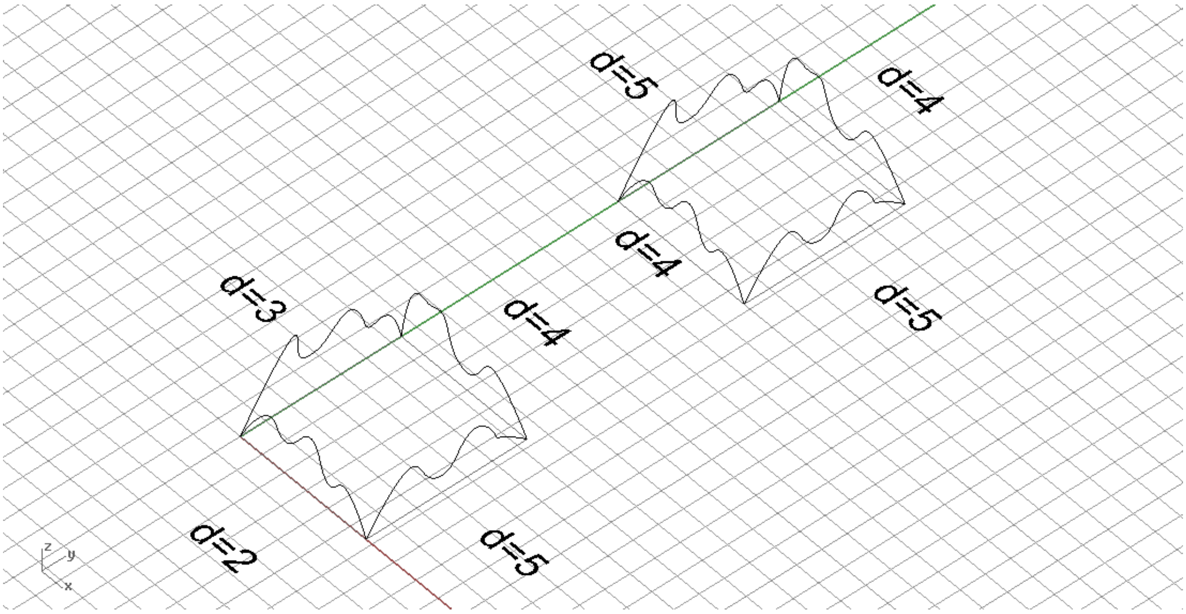


figure 92: Using lines with different degrees as the directrices of NURBS surfaces.

If a NURBS surface is generated by curves of different degrees in one of the parametric directions, the highest degree of surface generation in that direction is generally assumed. In the two previous figures, this fact is illustrated. The input lines for the NURBS generation were degree 2 and 4 in one of the parametric directions, and 3 and 5 in the other parametric direction. Thus, in the first direction, degree 4 was assumed for the first parametric direction and degree 5 for the other parametric direction. This is because it is possible to increase the degree of a curve without changing its shape, but not the other way around.

3.3. Different logics to generate surfaces

Modelling logics can vary. At this point we present some modelling logics implemented in Rhinoceros software. However, other applications may implement other logics. A cross reading of this point is recommended with that given in the tables on pages 26 and 39 for surface classification. The various logics contain variants. Here only the general ideas are presented trying to convey how the functions implemented in that application can be used directly for the surface representation.

3.3.1. Polygons

Polygons can be generated in various ways. A rectangle can be drawn using the *Plane* function, any n-sided polygon can be generated by defining the planar curves that delimit it (funtion *PlanarSrf*).

3.3.2. Surfaces of revolution (*Revolve* function)

About this type of surfaces, remember what was said in paragraph 3.2.2.

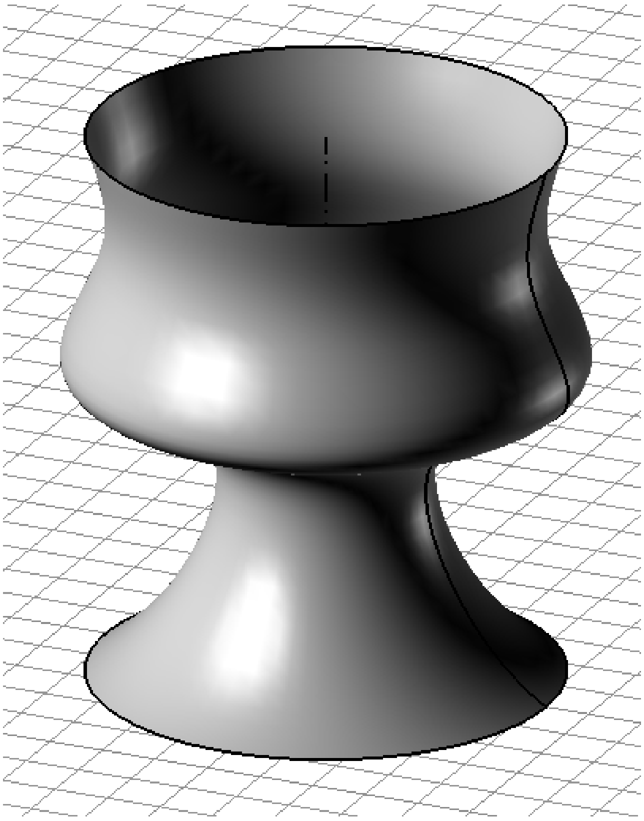


figure 93: Surface modelled in Rhinoceros with the Revolve function.

3.3.3. Other "revolution" surfaces (*RailRevolve* function)

These surfaces are based on the idea of rotation, but they are not truly surfaces of revolution. The generatrix rotates around an axis, however it deforms in this rotation as a function of the distances of the points of a new line (rail) in relation to the axis. These successive deformations are, in practice, affine transformations of the generatrix in orthogonal axis directions.

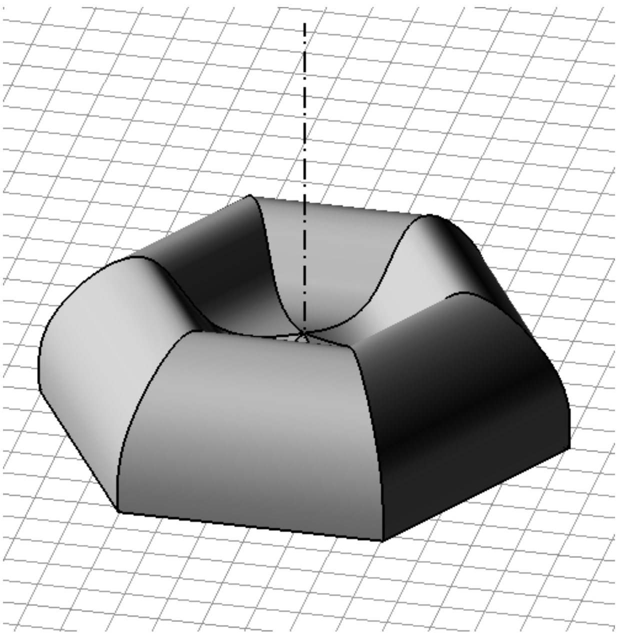


figure 94: Surface modelled in Rhinoceros with the RailRevolve function.

3.3.4. Translational surfaces (Sweep1 function)

These surfaces are based on the displacement of the generating line along a guideline with the restriction of maintaining the angle between them throughout the movement.

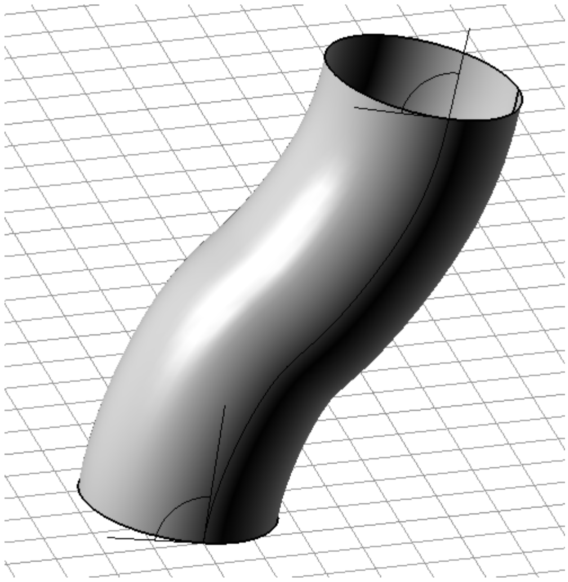


figure 95: Surface modelled in Rhinoceros with the Sweep1 function.

3.3.5. 3.3.4. Translational surfaces (Sweep2 function)

In this example (figure bellow) we have been given two horizontal guidelines (rails) along which the generatrix has moved, and the extreme positions of a generatrix, which being different, assume that the generatrix is deformable. The generatrix begins by being a semicircle arc to end in a semi-ellipse arc. In the intermediate positions, the ratio between minor axis and major axis varies between those of the two extremes.

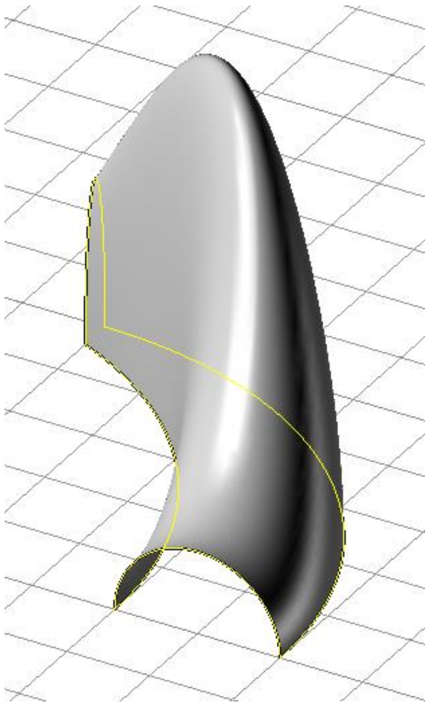


figure 96: Surface modelled in Rhinoceros with the Sweep2 function.

You can define more than two positions (and geometries) for the generatrix that will condition the surface generation to suit it.

3.3.6. Extrusion surfacecs (Extrude functions)

The basic idea of an extrusion consists in the definition of a generatrix that moves according to a guideline (directrix). In this displacement the generatrix maintains the orientation.

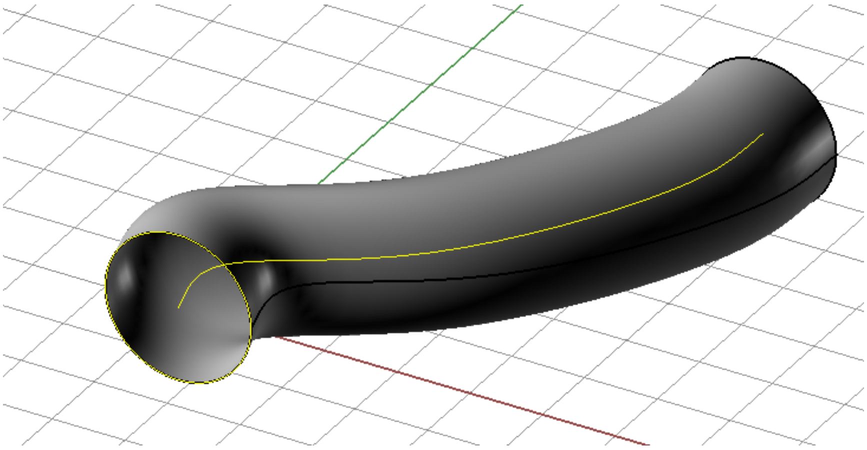


figure 97: Extrusion surface modelled in Rhinoceros.

There are, however, other types of extrusion, and the generatrix can change its size and proportions.

3.3.7. Surface generated by interpolating a sequence of curves (Loft function)

Considering several spatially given curves in a given sequence, the surface is generated by interpolation and fitting them according to specific criteria.

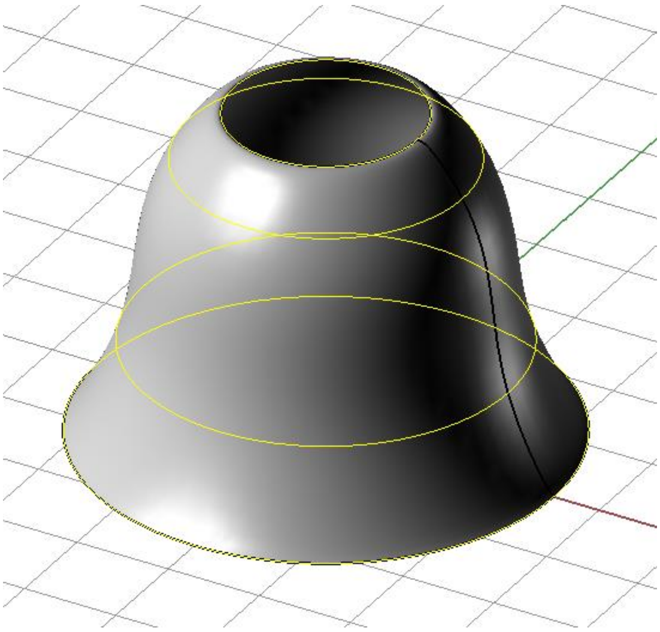


figure 98: Loft between a sequence of circles modelled in Rhinoceros.

3.3.8. Interpolating and fitting a spatial network of curves (NetworkSrt function)

As in the previous case, the surface is generated by interpolation and fitting. However, in this case, a network of space curves is given that inform about the "skeleton" of the surface.

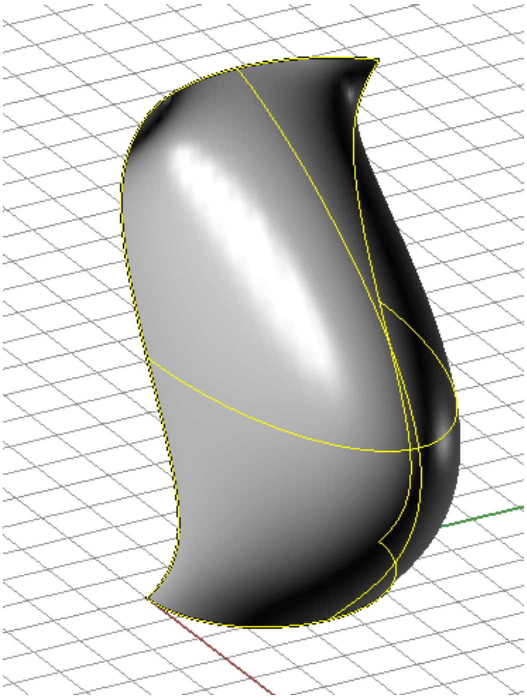


figure 99: Surface based in a network of curves modelled in Rhinoceros.

An identical way to generate surfaces is by defining a spatial point network (*SrtPtGrid* function). Each line of points corresponds in practice to a surface generating line in one of its parametric directions.

3.3.9. Generating a surface given four corner points (SrfPt function)

The conceptual surface that is generated this way is the hyperbolic paraboloid.

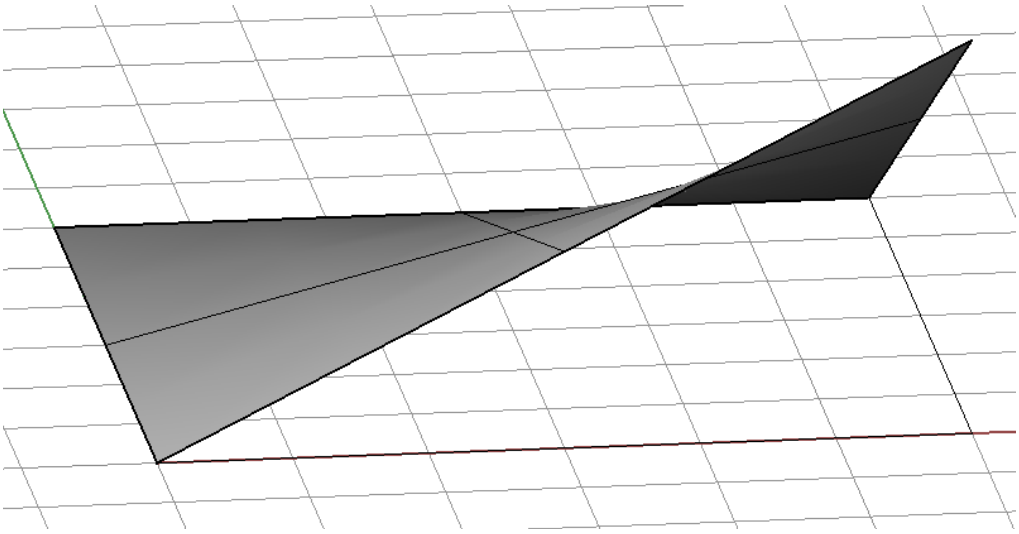


figure 100: Hyperbolic paraboloid modelled in Rhinoceros.

3.3.10. Generating a surface given the edge curves (EdgeSrf function)

In this case, the surface can be defined by 2, 3 or 4 edge lines. Ideally these should be boundary lines for better result control. In this case, the lines can also be other surface boundaries. And so, the tangency to these surfaces can be controlled.

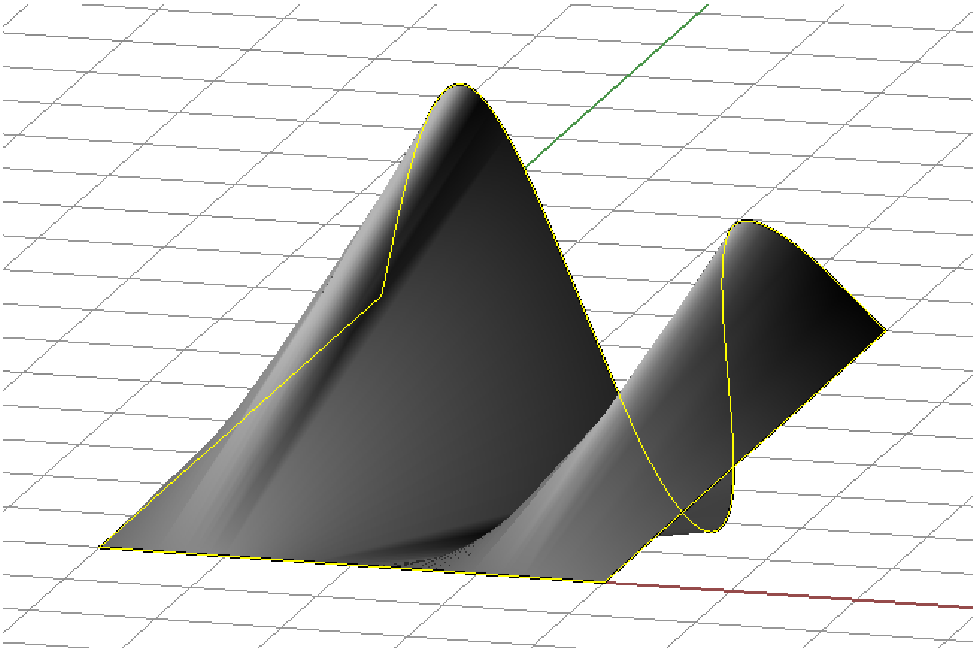


figure 101: Seurface modelled in Rhinoceros given 4 edge lines.

3.3.11. Surface passing and fitting a set of points, lines and meshes (Patch function)

In the example bellow, the surface was generated patching two horizontal rectangles at different heights and two points at different heights also. As it can be seen, the surface doesn´t perfectly fit the given elements.

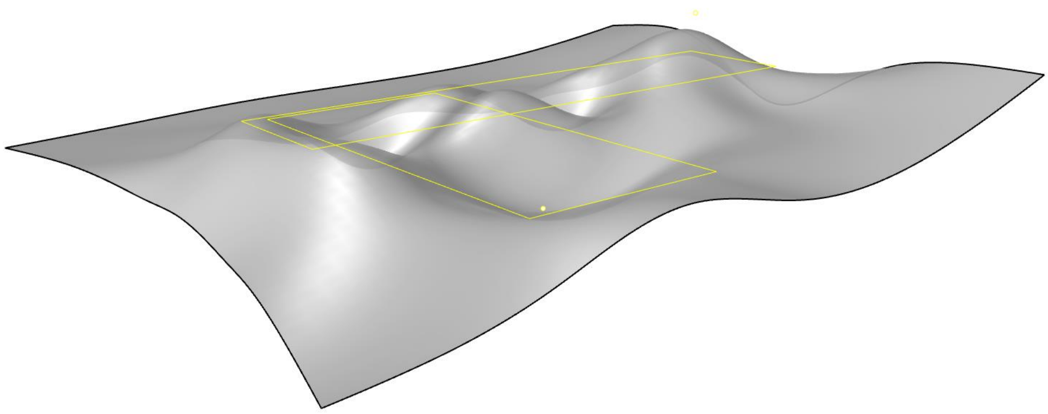


figure 102: Patch surface.

3.3.12. Surface generated by draping over given surfaces (Drape function)

The idea for generating this type of surface is to cover a set of surfaces given with a new surface as if they were covered with a cloth or elastic net. It is important to note that in Rhinoceros software this type of surface is always generated considering the current view. It is also important to note that surface depth control is a function of the nearest and the furthest point from the camera in the current view.

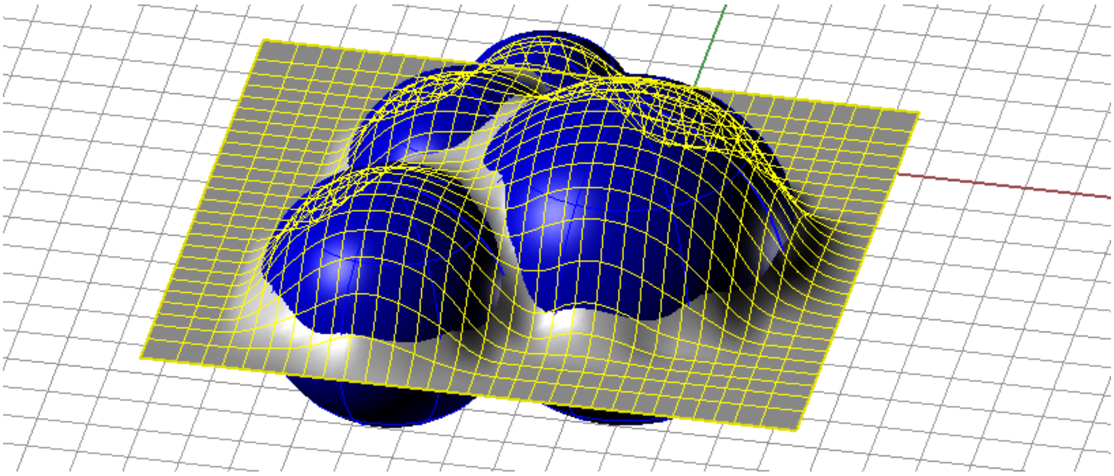


figure 103: Surface draped over a set of spheres.

3.3.13. Complex surfaces and composition of surfaces

With the resources given by computation, almost any surfaces can be represented. The following figure illustrates how, by properly choosing lines and points as guiding elements, we can represent a wide range of complex surfaces articulated to generate highly elaborated models.

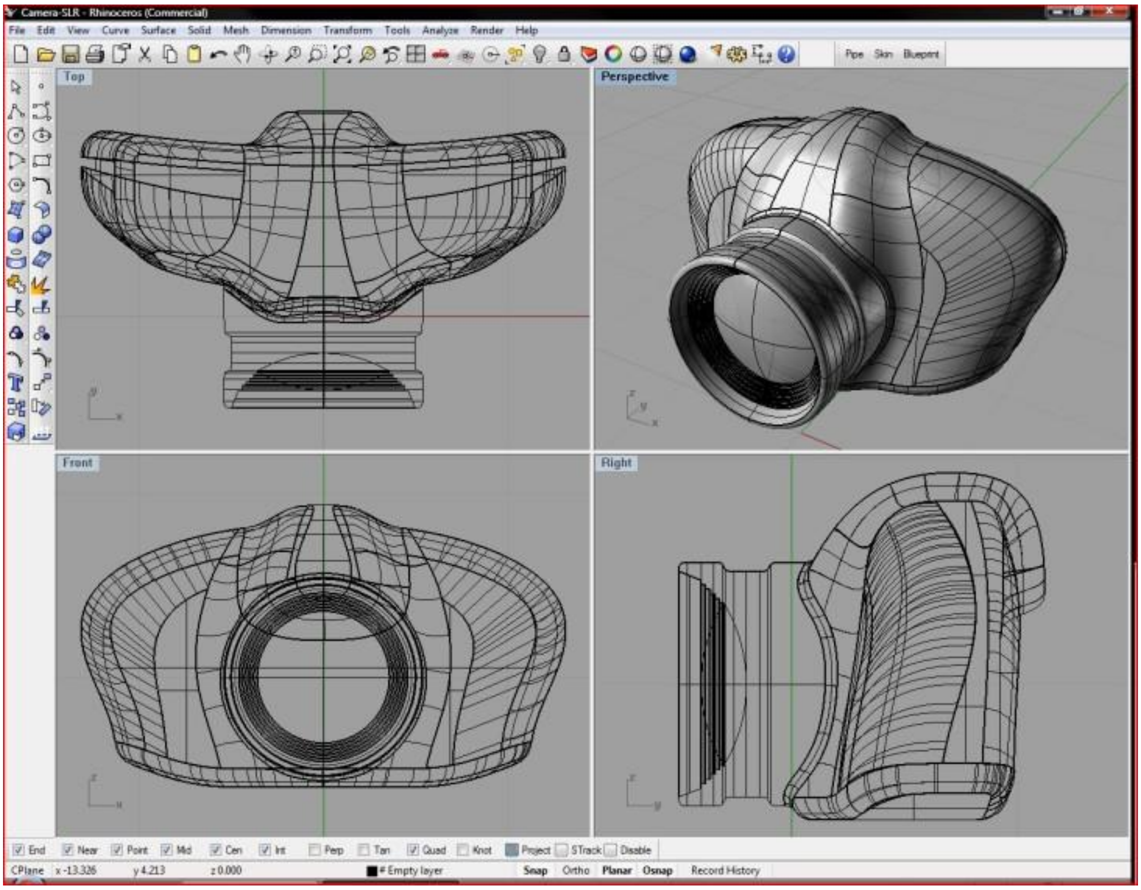


figure 104: A camera modelled in Rhinoceros (*in* http://www.3dprinter.net/rhino-3d-review).

3.4. Lines from surfaces

If, on the one hand, lines are what give rise to surfaces through their movement in space, it is also a fact that they can be generated from surfaces.

From a limited surface these limits can be extracted. From a NURBS surface, isoparametric lines can be extracted. Lines can be mapped to surfaces through different types of transformation (parallel projection, normal surface projection, parametric matching, etc.).

But the most popular way to generate lines across surfaces is through intersection, as illustrated in the following figure.

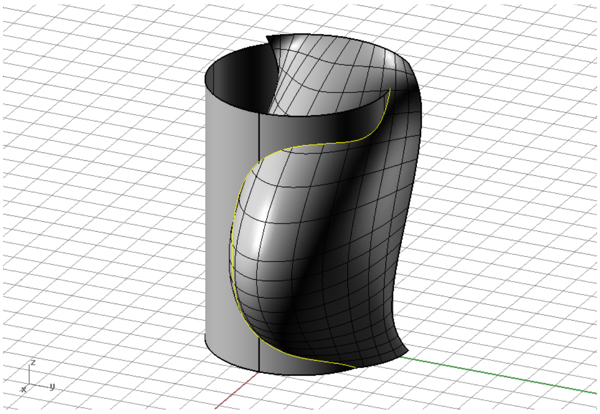


figure 105: A line generated at the intersection of two surfaces.

4. SOLIDS

Any surface configuration in space enclosing a volume can be conceptually associated with a solid. The edges of this solid (not necessarily straight) will be the intersecting lines of the various surfaces, and these lines will delimit the faces of the solid (not necessarily planar).

4.1. Boolean operations between solids

Boolean operations are of three types: a) union, b) subtraction, and c) intersection.

The union operation corresponds to considering everything that is common and not common to base solids.

The subtraction operation corresponds to the difference between the base solids, that is, what of the various base solids is common to the first, is subtracted from it.

The intersection operation corresponds to considering only the volume portion common to the base solids.

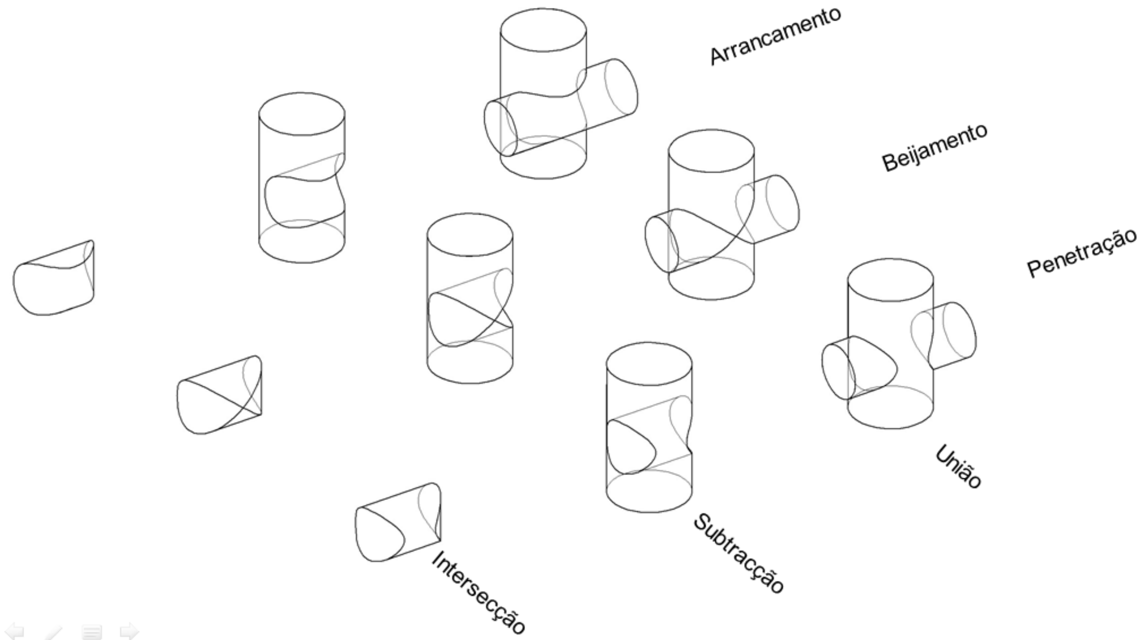


figure 106: Intersection types

In all cases, the edges of the final solids always result from the intersection of the base solid surfaces.

In addition to this classification of intersections, it is possible to consider others, as regards the characteristics of intersection lines between surfaces. We are particularly interested in highlighting a situation called PENETRATION with the intersection line having a singular point, as its control is more demanding. The condition that must be met for a double point on the intersection line between two surfaces is that there is a tangent plane common to both surfaces at that point.